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# Approximation Properties and Construction of Hermite Interpolants and Biorthogonal Multiwavelets[1]

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Multiwavelets are generated from refinable function vectors by using multiresolution analysis. In this paper we investigate the approximation properties of a multivariate refinable function vector associated with a general dilation matrix in terms of both the subdivision operator and the order of sum rules satisfied by the matrix refinement mask. Based on a fact about the sum rules of biorthogonal multiwavelets, a coset by coset (CBC) algorithm is presented to construct biorthogonal multiwavelets with arbitrary order of vanishing moments. More precisely, to obtain biorthogonal multiwavelets, we have to construct primal and dual masks. Given any primal matrix mask $a$ and a general dilation matrix $M$, the proposed CBC algorithm reduces the construction of all dual masks of $a$, which satisfy the sum rules of arbitrary order, to a problem of solving a well organized system of linear equations. We prove in a constructive way that for any given primal mask $a$ with a dilation matrix $M$ and for any positive integer $k$, we can always construct a dual mask $\tilde{a}$ of $a$ such that $\tilde{a}$ satisfies the sum rules of order $k$. In addition, we provide a general way for the construction of Hermite interpolatory matrix masks in the univariate setting with any dilation factors. From such Hermite interpolatory masks, smooth Hermite interpolants, including the well known cubic Hermite splines as a special case, are obtained and are used to construct biorthogonal multiwavelets. As an example, a $C^3$ Hermite interpolant with support $[-3, 3]$ is presented. Then we shall apply the CBC algorithm to such Hermite interpolatory masks to construct biorthogonal multiwavelets. Several examples of biorthogonal multiwavelets are provided to illustrate the general theory. In particular, a $C^1$ dual function vector with support $[-4, 4]$ of the cubic Hermite splines is given. © 2001 Academic Press

*Key Words:* biorthogonal multivariate multiwavelets; Hermite interpolants; Hermite interpolatory mask; accuracy order; sum rules; refinable function vectors.

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## 1. INTRODUCTION

In her celebrated paper [10], Daubechies constructed a family of compactly supported univariate orthogonal scaling functions and their corresponding orthogonal wavelets with the dilation factor 2. Since then wavelets with compact support have been widely and successfully used in various applications such as image compression and signal processing [11]. Each Daubechies wavelet is generated from one scaling function and therefore, is called a scalar wavelet.

Though orthogonal wavelets have many desired properties such as compact support, good frequency localization and vanishing moments, they lack symmetry as demonstrated by Daubechies in [11]. However, symmetry of wavelets is a much desired property in applications. Such a property is claimed to produce less visual artifacts than non-symmetric wavelets. To achieve symmetry, several generalizations of scalar orthogonal wavelets have been studied in the literature. For example, biorthogonal wavelets achieve symmetry where orthogonality is replaced with biorthogonality, and multiwavelets achieve both orthogonality and symmetry where one scaling function is replaced with several scaling functions (i.e., a scaling function vector). Wavelets in multidimensional spaces with a general dilation matrix have also been extensively investigated in recent years since in many applications we have to deal with higher dimensional data such as images.

Scalar biorthogonal wavelets in both univariate case and multivariate case have been extensively studied in the literature. See [5–7, 11, 16, 17, 20, 32, 41] and references therein for discussion on scalar biorthogonal wavelets. With symmetry and many other desired properties, scalar biorthogonal wavelets have been found to be more efficient and useful in many applications than the orthogonal ones [11].

To achieve symmetry, another approach is to adopt multiwavelets where a scaling function vector instead of a single scaling function is used. To compare with scalar wavelets, multiwavelets have several advantages such as shorter support and higher vanishing moments. The success of wavelets largely contributes to the short support and high vanishing moments which are competing objectives in the design of wavelets. That is, to obtain a wavelet with higher vanishing moments, it is necessary to enlarge its support. Such advantages of multiwavelets provide new opportunities and choices in the wavelet theory which are impossible to achieve by using scalar wavelets. With more flexible trade-off between high vanishing moments and short support, multiwavelets are particularly attractive in the construction of wavelets on a bounded domain [8] to deal with problems arising from a finite domain with boundary conditions and are expected to be useful in many applications such as numerical solutions to partial differential equations [8]. As a generalization of the scalar wavelets, it is also of interest in its own right to investigate multiwavelets. The advantages of multiwavelets and their promising features in applications have attracted a great deal of interest and effort in recent years to extensively study them. To only mention a few references here, see [8, 9, 12, 21–23, 30, 34, 36, 37, 40, 42] and references therein on discussion of various topics on multiwavelets and their applications. The generalization of scalar wavelets to multiwavelets is not trivial and the study of multiwavelets is much more complicated and involved than the study of the scalar wavelets which we shall see later.

Before proceeding further, let us introduce some notation. An $s \times s$ integer matrix $M$ is called a *dilation matrix* if $\lim_{n \to \infty} M^{-n} = 0$. That is, all the eigenvalues of a dilation matrix $M$ are greater than one in modulus. Throughout this paper, $M$ denotes a dilation matrix and $m := |\det M|$.

In this paper, we are concerned with the following *refinement equation*,

$$\phi = \sum_{\beta \in \mathbb{Z}^s} a(\beta)\, \phi(M \cdot - \beta), \tag{1.1}$$

where $\phi = (\phi_1, ..., \phi_r)^T$ is a $r \times 1$ vector of functions, called a *refinable function vector*, and $a$ is a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}^s$, called the (*matrix refinement*) *mask*. When $r = 1$, $\phi$ is called a *scalar refinable function* and $a$ is called a *scalar refinement mask*. Throughout this paper, all masks and refinable function vectors are assumed to be compactly supported.

By $J^a(0)$ we denote the following matrix associated with a mask $a$ as

$$J^a(0) := \sum_{\beta \in \mathbb{Z}^s} a(\beta).$$

If $\phi_1, ..., \phi_r$ are functions in $L_1(\mathbb{R}^s)$ with stable shifts and $\phi = (\phi_1, ..., \phi_r)^T$ satisfies the refinement equation (1.1) with a mask $a$, then it was proved by Dahmen and Micchelli [9] that

$$J^a(0) \text{ has a simple eigenvalue } m \text{ and all other eigenvalues in modulus} < m. \tag{1.2}$$

Conversely, if $J^a(0)$ satisfies the condition (1.2), it was proved by Heil and Collella [22] and Cabrelli *et al.* [3] that there exists a unique vector $\phi$ of compactly supported distributions such that $\phi$ satisfies (1.1) and $J^a(0)\, \hat{\phi}(0) = m\hat{\phi}(0)$ with $\|\hat{\phi}(0)\|_2 = 1$. We call such solution the *normalized distributional solution* of (1.1) and throughout this paper we denote such normalized solution of (1.1) with mask $a$ by $\phi^a$. If $\phi$ is another distribution solution of (1.1), then we must have $\phi = c\phi^a$ for some constant $c$.

On the one hand, multiwavelets provide more flexibility and new features which are not possible for scalar wavelets. On the other hand, in the multiwavelet case $r > 1$, each element in the mask of the refinement equation becomes an $r \times r$ matrix comparing with a scalar number in the scalar wavelet case. The change from a mask of scalar numbers to a mask of matrices makes the analysis and investigation of multiwavelets far more complicated and involved than its scalar counterpart. For example, even in the univariate case, the sum rule and vanishing moment conditions of a matrix mask are much more complicated than the scalar case, see [2, 4, 9, 23, 29, 31, 36, 39]. The involvement of matrices in the mask makes the construction of multiwavelets with certain vanishing moments much more challenging than its scalar counterpart. To our best knowledge, no systematic method is proposed in the current literature to construct biorthogonal multiwavelets with arbitrary order of sum rules even for the simplest case—the univariate setting with the dilation factor 2. Many known approaches and constructions in the scalar wavelet case do not apply to the multiwavelet case. Since both biorthogonal multivariate wavelets and multiwavelets are of great interest in both theory and applications, one aim of this paper is to investigate the biorthogonal multiwavelets in multidimensional spaces and to propose a method to construct them systematically.

Let $\ell(\mathbb{Z}^s)$ denote the linear space of all sequences on $\mathbb{Z}^s$ and $\ell_0(\mathbb{Z}^s)$ denote the linear space of all finitely supported sequences on $\mathbb{Z}^s$. For any positive integer $r$, by $(\ell(\mathbb{Z}^s))^{r\times r}$ we denote the linear space of all sequences of $r \times r$ matrices on $\mathbb{Z}^s$ and by $(\ell(\mathbb{Z}^s))^r$ we denote the linear space of all sequences of $r \times 1$ vectors on $\mathbb{Z}^s$. Similarly, we define $(\ell_0(\mathbb{Z}^s))^{r\times r}$ and $(\ell_0(\mathbb{Z}^s))^r$. Given any compactly supported distribution vector $f = (f_1, ..., f_r)^T$, we define the following linear operator $c_f$ on $(\ell(\mathbb{Z}^s))^r$ by

$$c_f(\lambda) := \sum_{\beta \in \mathbb{Z}^s} \lambda(\beta)^T f(\cdot - \beta), \qquad \lambda \in (\ell(\mathbb{Z}^s))^r, \tag{1.3}$$

where $A^T$ denotes the transpose of a matrix $A$. The shifts of $f$ are said to be *linearly independent* if $c_f(\lambda) = 0$ for $\lambda$ in $(\ell(\mathbb{Z}^s))^r$ implies $\lambda = 0$. It was proved by Jia and Micchelli [28] that the shifts of a compactly supported distribution vector $f = (f_1, ..., f_r)^T$ are linearly independent if and only if the sequences $(\hat{f}_j(\xi + 2\pi\beta))_{\beta \in \mathbb{Z}^s}$, $j = 1, ..., r$ are linearly independent for all $\xi \in \mathbb{C}^s$. The shifts of $f$ are *stable* if the sequences $(\hat{f}_j(\xi + 2\pi\beta))_{\beta \in \mathbb{Z}^s}$, $j = 1, ..., r$ are linearly independent for all $\xi \in \mathbb{R}^s$. Therefore, if the shifts of $\phi^a$ are stable or linearly independent, then (1.2) holds true.

Before proceeding further, let us introduce some notation. Recall that $M$ denotes a dilation matrix. Let $\Omega_M$ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s / M\mathbb{Z}^s$. Without loss of generality, we assume that

$0 \in \Omega_M$. For any mask $a$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$, we shall use the following notation throughout this paper

$$J_\varepsilon^a(\mu) := \sum_{\beta\in\mathbb{Z}^s} a(\varepsilon + M\beta)(M^{-1}\varepsilon + \beta)^\mu/\mu!, \qquad \mu \in \mathbb{Z}_+^s, \quad \varepsilon \in \Omega_M \tag{1.4}$$

and

$$J^a(\mu) := \sum_{\beta\in\mathbb{Z}^s} a(\beta)(M^{-1}\beta)^\mu/\mu! = \sum_{\varepsilon\in\Omega_M} J_\varepsilon^a(\mu), \qquad \mu \in \mathbb{Z}_+^s, \tag{1.5}$$

where

$$\mathbb{Z}_+^s := \{(\beta_1, ..., \beta_s) \in \mathbb{Z}^s : \beta_j \geqslant 0, \ j = 1, ..., s\}$$

and for any $\beta = (\beta_1, ..., \beta_s) \in \mathbb{Z}^s$ and $\mu = (\mu_1, ..., \mu_s) \in \mathbb{Z}_+^s$, $\beta^\mu := \beta_1^{\mu_1} \cdots \beta_s^{\mu_s}$, $\mu! := \mu_1! \cdots \mu_s!$ and $|\beta| := |\beta_1| + \cdots + |\beta_s|$. For any $\nu = (\nu_1, ..., \nu_s)$, $\mu = (\mu_1, ..., \mu_s) \in \mathbb{Z}_+^s$, we say that $\nu \leqslant \mu$ if $\nu_j \leqslant \mu_j$ for all $j = 1, ..., s$, and we say that $\nu < \mu$ if $\nu \leqslant \mu$ and $\nu \neq \mu$. For any mask $a$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$, we say that the mask $a$ with a dilation matrix $M$ satisfies the *sum rules* of order $k$ if there exists a set of $r \times 1$ vectors $\{y_\mu : \mu \in \mathbb{Z}_+^s, \ |\mu| < k\}$ with $y_0 \neq 0$ such that

$$\sum_{0\leqslant\nu\leqslant\mu} (-1)^{|\nu|} J_\varepsilon^a(\nu)^T y_{\mu-\nu} = \sum_{|\nu|=|\mu|} m_\nu^\mu y_\nu \qquad \forall \mu \in \mathbb{Z}_+^s, \quad |\mu| < k, \quad \varepsilon \in \Omega_M, \tag{1.6}$$

where the numbers $m_\nu^\mu$ are uniquely determined by

$$\frac{(M^{-1}x)^\mu}{\mu!} = \sum_{|\nu|=|\mu|} m_\nu^\mu \frac{x^\nu}{\nu!}, \qquad x \in \mathbb{R}^s. \tag{1.7}$$

Given a vector $\phi = (\phi_1, ..., \phi_r)^T$ of compactly supported distributions on $\mathbb{R}^s$, let $S(\phi) = \{c_\phi(\lambda) : \lambda \in (\ell(\mathbb{Z}^s))^r\}$ where the linear operator $c_\phi$ is defined in (1.3). Following Heil *et al.* [23], we say that $\phi$ has *accuracy order* $k$ if $\Pi_{k-1} \subseteq S(\phi)$ where $\Pi_{k-1}$ denotes the set of all polynomials of total degree less than $k$. We also agree that $\Pi_{-1} = \{0\}$. Accuracy order of $\phi$ has a close relation with both the approximation order provided by $\phi$ and the well known Strang–Fix conditions on $\phi$. See Jia [27] for such concepts and related results.

When $\phi^a$ is a refinable function vector with a mask $a$ and a dilation matrix $M$, under the assumption that the shifts of $\phi^a$ are linearly independent, Cabrelli *et al.* [2, 3] characterized the accuracy order of $\phi^a$ in terms of the order of sum rules satisfied by the mask $a$. Also see [1, 4, 23, 24, 29, 31, 36] and references therein for discussion on accuracy order under various conditions. In Section 2, we shall summarize and unify the discussion on accuracy order of a refinable distribution vector in the literature and provide a characterization (see Theorem 2.4) of the accuracy order of $\phi^a$ in terms of both the subdivision operator and sum rules under a very mild condition.

Accuracy order of a refinable function vector is also closely related to the concept of vanishing moments of biorthogonal multiwavelets (see [11]). The success of a wavelet basis largely lies in the short support and high accuracy order of a refinable function vector. It is also known that if a refinable function vector $\phi^a \in C^k$ has linearly independent shifts, then it is necessary that $\phi^a$ has accuracy order $k+1$. A function vector $\phi$ in $(L_2(\mathbb{R}^s))^r$ is called a *primal function vector* if $\phi$ satisfies the refinement equation (1.1) with a mask $a$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ and the shifts of $\phi$ are linearly independent. *A dual function vector* $\tilde{\phi}$ of $\phi$ is a function vector in $(L_2(\mathbb{R}^s))^r$ such that $\tilde{\phi}$ satisfies the refinement equation (1.1) with a mask $\tilde{a}$ and

$$\int_{\mathbb{R}^s} \overline{\phi(x)}\, \tilde{\phi}(x+\beta)^T\, dx = \delta(\beta)\, I_r \qquad \forall \beta \in \mathbb{Z}^s, \tag{1.8}$$

where $I_r$ denotes the $r\times r$ identity matrix, and $\delta(0)=1$, $\delta(\beta)=0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$. Clearly, if $\phi$ and $\tilde{\phi}$ satisfy the conditions in (1.8), then the shifts of $\phi$ and $\tilde{\phi}$ are linearly independent, respectively. A necessary condition for $\phi$ and $\tilde{\phi}$ to satisfy the conditions in (1.8) is the following well known discrete biorthogonal relations:

$$\sum_{\beta\in\mathbb{Z}^s} \overline{a(\beta)}\, \tilde{a}(\beta+M\alpha)^T = m\delta(\alpha)\, I_r \qquad \forall \alpha \in \mathbb{Z}^s. \tag{1.9}$$

If a mask $a$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ satisfies (1.2) and there exists a sequence $\tilde{a}$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ such that the conditions in (1.9) are satisfied, then we say that $a$ is a *primal mask* and any such mask $\tilde{a}$ will be called a *dual mask* of $a$. Dahmen and Micchelli proved in [9] that if $a$ is a primal mask with $\tilde{a}$ being a dual mask of $a$, then $\phi^a$ is a primal function vector with $\phi^{\tilde{a}}$ being a dual function vector of $\phi^a$ if and only if the subdivision schemes associated with $a$ and $\tilde{a}$ converge in the $L_2$ norm, respectively.

Given a primal mask $a$, to construct a dual mask of $a$, we need to solve a system of linear equations given in (1.9). In the current literature, the lifting scheme is known to be a good method for constructing a dual mask of any given primal mask. For discussion on lifting scheme, the reader is referred to [14, 32, 41]. After submitting this paper, we became aware of the preprint of F. Keinert titled “Raising multiwavelet approximation order through lifting” on discussion of lifting scheme in the univariate setting. However, by such lifting scheme, to increase the order of sum rules satisfied by the dual mask, both the primal and dual masks have to be changed simultaneously.

We point out that in the univariate setting, Plonka's factorization technique of a matrix symbol is very useful in studying refinable function vectors [34, 36] and was used by Plonka and Strela in [37] to construct smooth refinable function vectors. This factorization technique was also used by Strela in [40] to construct univariate biorthogonal multiwavelets. However, the factorization technique does not apply to the higher dimensions and the biorthogonal multiwavelets constructed by the factorization technique in [40] have very long support.

As we mentioned before, the vanishing moments of a biorthogonal multiwavelet are important in both applications and construction of smooth biorthogonal multiwavelets. See [5, 11] for discussion on vanishing moments and their relation to sum rules. For a primal mask $a$, the lifting scheme can be used to solve the discrete biorthogonal relation (1.9), which is a system of linear equations, to get a dual mask $\tilde{a}$ of $a$. To achieve high vanishing moments of the resulting biorthogonal multiwavelet, the dual mask $\tilde{a}$ must satisfy the sum rules of high order. Even in the simplest case $s=1$ and $M=(2)$, it is not easy to use the definition of sum rules given in (1.6) to achieve desired order of sum rules satisfied by $\tilde{a}$. The reason is that when $r>1$, to obtain a dual mask $\tilde{a}$ of a given primal mask $a$, even the equations in the biorthogonal conditions (1.9) are linear, the equations given in (1.6) for sum rules are no longer linear equations since in general the vectors $y_\nu$ in (1.6) are determined by $\tilde{a}$. Due to such difficulty, many methods on construction of scalar biorthogonal wavelets no longer hold in the multiwavelet case and not many examples of biorthogonal multiwavelets are available in the literature. For example, univariate multiwavelets were reported by Donovan *et al.* [12], Dahmen *et al.* [8], He and Lai [21], and other examples were given in [6, 14, 30, 40]. One purpose of this paper is to try to overcome such difficulty. In the scalar case, a coset by coset (CBC) algorithm was proposed by Han in [17] to construct scalar biorthogonal wavelets with arbitrary vanishing moments. In this paper, we shall generalize the CBC algorithm in [17] to the multiwavelet case to overcome the above mentioned difficulty. For the advantages of the CBC algorithm over other known methods on construction of scalar biorthogonal wavelets, the reader is referred to [5, 17, 20].

In Section 3, we shall follow the line developed in [5, 17, 20] to discuss how to construct biorthogonal multiwavelets in the most general case. We propose a general coset by coset (CBC) algorithm in Section 3. Such CBC algorithm reduces the construction of all dual masks, which satisfy the sum rules of arbitrary order, of a given primal mask to the problem of solving a well organized system of linear equations. Based on such algorithm, we shall demonstrate that for any given primal mask $a$ with a dilation matrix $M$ and for any positive integer $k$, we can construct a dual mask $\tilde{a}$ of $a$ such that $\tilde{a}$ satisfies the sum rules of order $k$. In Section 4, we construct a special family of primal masks—Hermite interpolatory masks in the univariate setting with any dilation factor. The resulting refinable function vectors are Hermite interpolants which are useful in curve design in computer aided geometric design [13]. As an example, a $C^3$ Hermite interpolant is constructed with support $[-3, 3]$. In particular, such construction of Hermite interpolatory masks includes the cubic Hermite splines as a special case. Several new examples of biorthogonal multiwavelets are provided to illustrate the general theory developed in this paper. In particular, a $C^1$ dual function vector of the piecewise Hermite cubics is given and is supported on $[-4, 4]$. Finally, by the CBC algorithm, we construct a continuous dual function vector for the primal function vector which is a piecewise polynomial B-spline of order 6 with double knots in Plonka and Strela [37]. Such primal function vector belongs to $C^{4-\eta}$ for any $\eta > 0$, has accuracy order 6 and has support $[0, 3]$.

A short outline of the paper is as follows. In Section 2, we discuss accuracy order of refinable distribution vectors under a very mild condition. In Section 3, we propose a CBC algorithm and we prove that for any primal mask with a dilation matrix, a dual mask with any preassigned order of sum rules can be constructed by such CBC algorithm. Finally, in Section 4, we construct a family of Hermite interpolatory masks and several examples of biorthogonal multiwavelets are presented to illustrate the general theory.

## 2. ACCURACY ORDER OF REFINABLE DISTRIBUTION VECTORS

In this section, under a very mild condition we shall investigate the accuracy order of a refinable distribution vector in terms of both the subdivision operator and the sum rules. Some results in this section are essentially known in the literature under various assumption. Here we shall discuss accuracy order under a very mild condition on the normalized distributional solution to (1.1) and therefore, our result in this section includes most of the results on accuracy order in [1, 4, 23, 24, 29, 31, 36] as special cases. Following the line developed in Jia [26], we shall give a short discussion on accuracy order without detailed proofs. Detailed treatment on accuracy order under various conditions can be found in [1, 4, 23, 24, 29, 31, 36].

Let us introduce some notation and several auxiliary results here. Given a mask $a$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$, the subdivision operator $S_a$ associated with the mask $a$ and a dilation matrix $M$ is defined by

$$S_a\lambda(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)^T \lambda(\beta), \qquad \alpha\in\mathbb{Z}^s,\ \lambda\in(\ell(\mathbb{Z}^s))^r. \tag{2.1}$$

By $\Pi_k^r$ we denote the set of $r\times 1$ polynomial vectors with each component being a polynomial of degree at most $k$, and $\Pi^r := \bigcup_{k=0}^{\infty} \Pi_k^r$. For any $p\in\Pi^r$, we shall use $p$ to denote both the polynomial $p(x)$ and the sequence $(p(\alpha))_{\alpha\in\mathbb{Z}^s}$ since they can be easily distinguished in the context.

LEMMA 2.1. *Let $p\in\Pi^r|_{\mathbb{Z}^s}$. Then $S_a p\in\Pi^r|_{\mathbb{Z}^s}$ if and only if*

$$\sum_{\beta\in\mathbb{Z}^s} a(\varepsilon+M\beta)^T\, p(x-M^{-1}\varepsilon-\beta)$$

$$=\sum_{\beta\in\mathbb{Z}^s} a(M\beta)^T\, p(x-\beta) \qquad \forall \varepsilon\in\Omega_M,\ x\in\mathbb{R}^s.$$

*If this is the case, then $S_a p(x)=\sum_{\beta\in\mathbb{Z}^s} a(\varepsilon+M\beta)^T\, p(M^{-1}x-M^{-1}\varepsilon-\beta)$ for any $\varepsilon\in\Omega_M$.*

By $D_j$ we denote the partial derivative with respect to the $j$th unit coordinate. For any $\mu=(\mu_1, ..., \mu_s)\in\mathbb{Z}_+^s$, $D^\mu$ denotes the differential operator $D_1^{\mu_1}\cdots D_s^{\mu_s}$. Moreover, we write $D=(D_1, ..., D_s)^T$.

For any polynomial $p\in\Pi^r$, we shall use the convention

$$p(x-iD)^T := \sum_{\mu\in\mathbb{Z}_+^s} p^{(\mu)}(x)^T\,(-iD)^\mu/\mu!,$$

where $i$ is the imaginary unit such that $i^2=-1$. For a mask $a$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$, let

$$H^a(\xi) := \sum_{\beta\in\mathbb{Z}^s} a(\beta)\, e^{-i\beta\cdot\xi}/m, \qquad \xi\in\mathbb{R}^s$$

and define

$$H_k^a(\xi) := H^a((M^T)^{-1}\xi)\cdots H^a((M^T)^{-k}\xi), \qquad k\in\mathbb{N}.$$

By computation, it is easy to verify the following result.

PROPOSITION 2.2. *If $p\in\Pi^r$, then $S_a p\in\Pi^r$ if and only if*

$$p(x-iD)^T\, H_1^a(2\pi\beta_0)=0 \qquad \forall \beta_0\in\mathbb{Z}^s\backslash M^T\mathbb{Z}^s. \tag{2.2}$$

*Moreover, if $p\in\Pi^r$ and $S_a p\in\Pi^r$, then $p(x-iD)^T\, H_1^a(2\pi M^T\beta)=(S_a p)(Mx)^T$ for all $\beta\in\mathbb{Z}^s$, and for any $k>1$,*

$$p(x-iD)^T\, H_k^a(2\pi M^T\beta)=(S_a p)(Mx-iD)^T\, H_{k-1}^a(2\pi\beta) \qquad \forall\beta\in\mathbb{Z}^s.$$

For any mask $a$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ and any positive integer $k$, we define a subspace $\mathscr{P}_k^a$ of $\Pi_k^r$ as

$$\mathscr{P}_k^a := \{p \in \Pi_k^r : S_a^j p \in \Pi_k^r|_{\mathbb{Z}^s}\ \forall j \in \mathbb{N}\}. \tag{2.3}$$

It is evident that $\mathscr{P}_k^a$ is invariant under the subdivision operator $S_a$ defined in (2.1). By using the results in [4, 26, 27], we have the following result.

THEOREM 2.3. *If $p \in \mathscr{P}_k^a$, then $c_{\phi^a}(p) \in \Pi_k$ where $\mathscr{P}_k^a$ is defined in (2.3) and the operator $c_{\phi^a}$ is defined in (1.3). In particular,*

$$\begin{aligned} c_{\phi^a}(p)(x) &= \sum_{\beta \in \mathbb{Z}^s} p(\beta)^T \phi^a(x-\beta) = p(x-iD)^T \hat{\phi}^a(0) \\ &= \sum_{\mu \in \mathbb{Z}_+^s} \frac{1}{\mu!} p^{(\mu)}(x)^T (-iD)\, \hat{\phi}^a(0). \end{aligned}$$

*Conversely, if the sequences $(\hat{\phi}_j(2\pi(M^T)^{-1}\varepsilon + 2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $j=1, ..., r$ are linearly independent for all $\varepsilon \in \Omega_{M^T}$ where $\phi^a = (\phi_1, ..., \phi_r)^T$ is the normalized solution of (1.1) with the mask a, then for any $q \in \Pi_k \cap S(\phi^a)$, there exists a unique $p \in \mathscr{P}_k^a$ such that $c_{\phi^a}(p) = q$.*

The result on accuracy order of a refinable distribution vector is the following.

THEOREM 2.4. *Let a be a mask in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ and satisfy the condition (1.2) with a dilation matrix M. Let $\phi^a = (\phi_1, ..., \phi_r)^T$ be the normalized distributional solution of the refinement equation (1.1) with the mask a and the dilation matrix M. Suppose that the sequences $(\hat{\phi}_j(2\pi(M^T)^{-1}\varepsilon + 2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $j=1, ..., r$ are linearly independent for all $\varepsilon \in \Omega_{M^T}$. Then the following statements are equivalent:*

(a) *$\phi^a$ has accuracy order k;*

(b) $\dim \mathscr{P}_{k-1}^a = \dim \Pi_{k-1}$ *where $\mathscr{P}_{k-1}^a$ is defined in (2.3);*

(c) *The mapping $c_{\phi^a}|_{\mathscr{P}_{k-1}^a} : \mathscr{P}_{k-1}^a|_{\mathbb{Z}^s} \to \Pi_{k-1}$ is onto where $c_{\phi^a}$ is defined in (1.3);*

(d) *a satisfies the sum rules of order k.*

*Moreover, if a satisfies the sum rules of order k given in (1.6) with $\{y_\mu : \mu \in \mathbb{Z}_+^s, |\mu| < k\}$ and $y_0^T \hat{\phi}^a(0) = 1$, then*

$$\frac{x^\mu}{\mu!} = \sum_{0\leqslant\nu\leqslant\mu} \sum_{\beta\in\mathbb{Z}^s} \frac{\beta^\nu}{\nu!} y_{\mu-\nu}^T \phi^a(x-\beta) \qquad \forall\ |\mu| < k, \mu \in \mathbb{Z}_+^s.$$

*Proof.* From Theorem 2.3, we see that for any positive integer $k$, the mapping $c_{\phi^a}|_{\mathscr{P}^a_{k-1}}\colon \mathscr{P}^a_{k-1} \to \Pi_{k-1}$ is well defined. Under the assumption that the sequences $(\hat{\phi}_j(2\pi\beta))_{\beta\in\mathbb{Z}^s}$, $j=1, ..., r$ are linearly independent, by [27, Lemma 8.2], $c_{\phi^a}|_{\mathscr{P}^a_{k-1}}$ is one-to-one. Therefore, by Theorem 2.3, $\phi^a$ has accuracy order $k$ if and only if $c_{\phi^a}|_{\mathscr{P}^a_{k-1}}$ is one-to-one and onto which is also equivalent to $\dim \mathscr{P}^a_{k-1} = \dim \Pi_{k-1}$. Thus, we proved that (a), (b), and (c) are equivalent.

Since $c_{\phi^a}|_{\mathscr{P}^a_{k-1}}$ is a one-to-one and onto mapping between $\mathscr{P}^a_{k-1}$ and $\Pi_{k-1}$, the inverse mapping of $c_{\phi^a}|_{\mathscr{P}^a_{k-1}}$ carries the structure of $\Pi_{k-1}$ into $\mathscr{P}^a_{k-1}$. Define

$$p_\mu := c_{\phi^a}^{-1}(x^\mu/\mu!), \qquad \mu\in\mathbb{Z}^s_+,\ |\mu|<k.$$

Since $c_{\phi^a}(D^\nu p_\mu) = D^\nu c_{\phi^a}(p_\mu)$ for all $\nu$, $\mu\in\mathbb{Z}^s_+$ with $|\mu|<k$, we may assume that

$$p_\mu = \sum_{0\leqslant\nu\leqslant\mu} y_{\mu-\nu}\frac{x^\nu}{\nu!} \qquad \forall\mu\in\mathbb{Z}^s_+,\ |\mu|<k, \tag{2.4}$$

for some $r\times 1$ vectors $y_\nu$, $|\nu|<k$. Note that $c_{\phi^a}(S_a p_\mu)(Mx) = c_{\phi^a}(p_\mu)(x) = x^\mu/\mu!$ We have

$$(S_a p_\mu)(x) = c_{\phi^a}^{-1}\left(\frac{(M^{-1}x)^\mu}{\mu!}\right) = c_{\phi^a}^{-1}\left(\sum_{|\nu|=|\mu|} m^\mu_\nu \frac{x^\nu}{\nu!}\right) = \sum_{|\nu|=|\mu|} m^\mu_\nu p_\nu(x),$$

where $m^\mu_\nu$ are given in (1.7). By Lemma 2.1, we have

$$\begin{aligned}\sum_{|\nu|=|\mu|} m^\mu_\nu p_\nu(Mx) &= S_a p_\mu(Mx)\\ &= \sum_{\beta\in\mathbb{Z}^s} a(\varepsilon+M\beta)^T p_\mu(x-M^{-1}\varepsilon-\beta) \qquad \forall\varepsilon\in\Omega_M.\end{aligned}$$

Setting $x=0$ in the above equality, we have (1.6). Hence, (c) implies (d).

To prove that (d) implies (a), we define $p_\mu$ as in (2.4). By the definition of sum rules and observing that $D^\nu(p_\mu) = p_{\mu-\nu}$ where by convention $p_\nu\equiv 0$ for $\nu\notin\mathbb{Z}^s_+$, it is not difficult to verify that $s_a p\in\Pi^r_k$ and $S_a p = \sum_{|\nu|=|\mu|} m^\mu_\nu p_\nu$ for all $|\mu|<k$. Thus $p_\mu\in\mathscr{P}^a_{k-1}$, and by Theorem 2.3, $c_{\phi^a}(p_\mu)\in\Pi_{k-1}$ for all $|\mu|<k$. Set $q_\mu := c_{\phi^a}(p_\mu)$. Then we have

$$q_\mu(M^{-1}x) = c_{\phi^a}(S_a p_\mu)(x) = \sum_{|\nu|=|\mu|} m^\mu_\nu q_\nu(x) \qquad \forall\mu\in\mathbb{Z}^s_+,\ |\mu|<k. \tag{2.5}$$

Note that $D^{\nu} q_{\mu} = q_{\mu-\nu}$ for all $\nu \in \mathbb{Z}_{+}^{s}$ (see [3, Theorem 3.2]). We may assume

$$q_{\mu}(x) = \sum_{0 \leqslant \nu \leqslant \mu} l_{\mu-\nu} \frac{x^{\nu}}{\nu!}, \qquad \mu \in \mathbb{Z}_{+}^{s}, |\mu| < k,$$

for some $l_{\mu} \in \mathbb{C}$, $\mu \in \mathbb{Z}_{+}^{s}$ with $|\mu| < k$. By induction, we have $l_{\mu} = 0$ for any $0 < |\mu| < k$. Hence, $q_{\mu} = l_0 x^{\mu}/\mu!$ for all $|\mu| < k$. By Theorem 2.3, $l_0 = y_0^T \hat{\phi}_0(0) \neq 0$ follows directly from (1.2) and $y_0 \neq 0$. ▮

In fact, $\mathscr{P}_k^a = \{p \in \Pi_k^r : S_a^j p \in \Pi_k^r \ \forall j = 1, ..., \dim \Pi_k^r\}$ since $\mathscr{P}_k^a \subseteq \Pi_k^r$. If $a$ satisfies the sum rules of order $k+1$, then $S_a|_{\mathscr{P}_k^a} = (c_{\phi^a}|_{\mathscr{P}_k^a})^{-1} \tau_M (c_{\phi^a}|_{\mathscr{P}_k^a})$ where $\tau_M \colon \Pi_k \to \Pi_k$ is given by $\tau_M(p)(x) = p(M^{-1}x)$. Therefore, the structure of $S_a$ restricted to the subspace $\mathscr{P}_k^a$ can be easily analyzed by using a much simpler operator $\tau_M$.

## 3. CONSTRUCTION OF DUAL MASKS WITH ARBITRARY ORDER OF SUM RULES

In this section, we shall discuss how to systematically construct dual masks with arbitrary order of sum rules for any given primal mask. More precisely, given a primal mask $a$, for any positive integer $k$, how to find all dual masks of $a$ such that the dual masks satisfy the sum rules of order $k$. Even in the scalar and multivariate case, this is not straightforward and there are a lot of literature discussing it, see [5, 7, 11, 16, 17, 25, 32, 41] and references therein. This problem is also called filter design in the language of engineering.

As for the multiwavelet case, even in the univariate setting, to our best knowledge, no systematic method is available in the current literature to deal with this problem. As a matter of fact, it took a relatively long time in the wavelet community to find a continuous dual scaling function vector of the well known cubic Hermite splines [8]. In this paper, we shall demonstrate that designing multiwavelets with arbitrary vanishing moments can be reduced to solve a system of well organized linear equations by using a CBC algorithm. As an application of this method, a $C^1$ dual scaling function vector with support $[-4, 4]$ of the cubic Hermite splines will be given in Section 4. More interesting is that a family of Hermite interpolatory masks will be constructed in Section 4 such that it includes the cubic Hermite splines as a special case. Several other examples of biorthogonal multiwavelets will also be provided in Section 4. Due to the length of this paper, we shall discuss the smoothness and convergence problems associated with the refinable function vectors of our construction elsewhere.

For any $j \in \mathbb{N} \cup \{0\}$, let $O_j$ be the ordered set of $\{\mu \in \mathbb{Z}^s_+ : |\mu| = j\}$ in the lexicographic order. That is, $(\nu_1, ..., \nu_s)$ is less than $(\mu_1, ..., \mu_s)$ in lexicographic order if $\nu_j = \mu_j$ for $j = 1, ..., i-1$ and $\nu_i < \mu_i$. By $\# O_j$ we denote the cardinality of the set $O_j$. The Kronecker product of two matrices $A = (a_{ij})_{1 \leqslant i \leqslant l, 1 \leqslant j \leqslant n}$ and $B$, written as $A \otimes B$, is defined to be the matrix

$$A \otimes B := \begin{bmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ a_{21} B & a_{22} B & \cdots & a_{2n} B \\ \vdots & \vdots & \ddots & \vdots \\ a_{l1} B & a_{l2} B & \cdots & a_{ln} B \end{bmatrix}.$$

The following result is crucial in the CBC algorithm and is a generalization of Theorem 6.1 in Han [17].

THEOREM 3.1. *Let $a$ and $\tilde{a}$ in $(\ell_0(\mathbb{Z}^s))^{r \times r}$ be two masks satisfying the discrete biorthogonal relation* (1.9) *with a dilation matrix $M$. Suppose that $J^a(0)$ satisfies the condition* (1.2) *and $\tilde{a}$ satisfies the sum rules of order $k$ for some positive integer $k$, i.e., there exist vectors $\tilde{y}_\mu$, $|\mu| < k$ in $\mathbb{C}^r$ with $\tilde{y}_0 \neq 0$ such that*

$$\sum_{0 \leqslant \nu \leqslant \mu} (-1)^{|\nu|} J^{\tilde{a}}_\varepsilon(\nu)^T \, \tilde{y}_{\mu-\nu} = \sum_{|\nu| = |\mu|} m^\mu_\nu \tilde{y}_\nu \qquad \forall \mu \in \mathbb{Z}^s_+, \quad |\mu| < k, \quad \varepsilon \in \Omega_M,$$

*where $m^\mu_\nu$ are given in* (1.7) *and $\Omega_M$ is a complete set of representatives of the distinct cosets of $\mathbb{Z}^s / M\mathbb{Z}^s$ with $0 \in \Omega_M$. Let $O_j$ be the ordered set of $\{\mu \in \mathbb{Z}^s_+ : |\mu| = j\}$ in the lexicographic order. Then $\tilde{y}_\mu$, $|\mu| < k$ are uniquely determined up to a scalar multiplier constant by the following recursive relation: $\overline{J^a(0)} \, \tilde{y}_0 = m \tilde{y}_0$ and for $j = 1, ..., k-1$, the vectors $\tilde{y}_\mu (\mu \in O_j)$ are determined by*

$$(\tilde{y}_\mu)_{\mu \in O_j} = [m I_{r(\# O_j)} - (m^\mu_\nu)_{\mu, \nu \in O_j} \otimes \overline{J^a(0)}]^{-1}$$
$$\times \left( \sum_{0 \leqslant \eta < \mu} \sum_{|\nu| = |\eta|} m^\eta_\nu \overline{J^a(\mu - \eta)} \, \tilde{y}_\nu \right)_{\mu \in O_j}.$$

*In particular, when $M = d I_s$ where $d$ is an integer, we have $\overline{J^a(0)} \, \tilde{y}_0 = d^s \tilde{y}_0$ and*

$$\tilde{y}_\mu = (d^{|\mu| + s} I_r - \overline{J^a(0)})^{-1} \sum_{0 \leqslant \nu < \mu} d^{|\mu - \nu|} \overline{J^a(\mu - \nu)} \, \tilde{y}_\nu,$$
$$0 < |\mu| < k, \quad \mu \in \mathbb{Z}^s_+.$$

*Proof.* From the discrete biorthogonal relation (1.9), we have

$$m \sum_{\alpha \in \mathbb{Z}^s} \delta(\alpha)\, I_r \frac{\alpha^\nu}{\nu!} = \sum_{\alpha \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} \overline{a(\beta)}\, \tilde{a}(\beta + M\alpha)^T \frac{\alpha^\nu}{\nu!}, \qquad \nu \in \mathbb{Z}^s_+.$$

Note that

$$\begin{aligned} \frac{\alpha^\nu}{\nu!} &= \frac{((M^{-1}\beta + \alpha) - M^{-1}\beta)^\nu}{\nu!} \\ &= \sum_{0 \leqslant \eta \leqslant \nu} (-1)^{|\eta|} \frac{(M^{-1}\beta)^\eta}{\eta!} \frac{(M^{-1}\beta + \alpha)^{\nu - \eta}}{(\nu - \eta)!}. \end{aligned}$$

Therefore,

$$\begin{aligned} m\delta(\nu)\, I_r &= \sum_{0 \leqslant \eta \leqslant \nu} \sum_{\alpha \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} (-1)^{|\eta|}\, \overline{a(\beta)} \frac{(M^{-1}\beta)^\eta}{\eta!} \\ &\quad \times \tilde{a}(\beta + M\alpha)^T \frac{(M^{-1}\beta + \alpha)^{\nu - \eta}}{(\nu - \eta)!} \\ &= \sum_{\varepsilon \in \Omega_M} \sum_{0 \leqslant \eta \leqslant \nu} (-1)^{|\eta|} \sum_{\beta \in \mathbb{Z}^s} \overline{a(\varepsilon + M\beta)} \frac{(M^{-1}\varepsilon + \beta)^\eta}{\eta!} J^{\tilde{a}}_\varepsilon(\nu - \eta)^T. \end{aligned}$$

Hence, we have

$$m\delta(\nu)\, I_r = \sum_{\varepsilon \in \Omega_M} \sum_{0 \leqslant \eta \leqslant \nu} (-1)^{|\eta|}\, \overline{J^a_\varepsilon(\eta)}\, J^{\tilde{a}}_\varepsilon(\nu - \eta)^T, \qquad \nu \in \mathbb{Z}^s_+. \tag{3.1}$$

For any $\nu,\ \mu \in \mathbb{Z}^s_+$ such that $|\mu| < k$ and $\nu \leqslant \mu$, multiplying $(-1)^{|\nu|}\, \tilde{y}_{\mu - \nu}$ with both sides of (3.1) and taking sum, we have

$$\begin{aligned} m\tilde{y}_\mu &= \sum_{0 \leqslant \nu \leqslant \mu} (-1)^{|\nu|}\, m\delta(\nu)\, \tilde{y}_{\mu - \nu} \\ &= \sum_{\varepsilon \in \Omega_M} \sum_{0 \leqslant \nu \leqslant \mu} \sum_{0 \leqslant \eta \leqslant \nu} (-1)^{|\nu - \eta|}\, \overline{J^a_\varepsilon(\eta)}\, J^{\tilde{a}}_\varepsilon(\nu - \eta)^T\, \tilde{y}_{\mu - \nu} \\ &= \sum_{\varepsilon \in \Omega_M} \sum_{0 \leqslant \eta \leqslant \mu} \sum_{\eta \leqslant \nu \leqslant \mu} (-1)^{|\nu - \eta|}\, \overline{J^a_\varepsilon(\eta)}\, J^{\tilde{a}}_\varepsilon(\nu - \eta)^T\, \tilde{y}_{\mu - \nu} \\ &= \sum_{\varepsilon \in \Omega_M} \sum_{0 \leqslant \eta \leqslant \mu} \overline{J^a_\varepsilon(\eta)} \sum_{0 \leqslant \nu \leqslant \mu - \eta} (-1)^{|\nu|}\, J^{\tilde{a}}_\varepsilon(\nu)^T\, \tilde{y}_{\mu - \eta - \nu}. \end{aligned}$$

Since $\tilde{a}$ satisfies the sum rules of order $k$, by (1.6) and the above equality, we have

$$
\begin{aligned}
m\tilde{y}_\mu &= \sum_{\varepsilon\in\Omega_M}\ \sum_{0\leqslant\eta\leqslant\mu} \overline{J^a_\varepsilon(\eta)} \sum_{|\nu|=|\mu-\eta|} m^{\mu-\eta}_\nu \tilde{y}_\nu \\
&= \sum_{0\leqslant\eta\leqslant\mu}\ \sum_{|\nu|=|\mu-\eta|} m^{\mu-\eta}_\nu \overline{J^a(\eta)}\ \tilde{y}_\nu = \sum_{0\leqslant\eta\leqslant\mu}\ \sum_{|\nu|=|\eta|} m^\eta_\nu \overline{J^a(\mu-\eta)}\ \tilde{y}_\nu.
\end{aligned}
$$

Thus, we deduce that

$$
m\tilde{y}_\mu - \sum_{|\nu|=|\mu|} m^\mu_\nu \overline{J^a(0)}\ \tilde{y}_\nu = \sum_{0\leqslant\eta<\mu}\ \sum_{|\nu|=|\eta|} m^\eta_\nu \overline{J^a(\mu-\eta)}\ \tilde{y}_\nu, \qquad \mu\in\mathbb{Z}^s_+,\quad |\mu|<k. \tag{3.2}
$$

Regarding $(\tilde{y}_\mu)_{\mu\in O_j}$ as an $r(\#O_j)\times 1$ column vector where $\#O_j$ is the cardinality of the set $O_j$, Eq. (3.2) can be rewritten in the matrix form

$$
\begin{aligned}
&[mI_{r(\#O_j)} - (m^\mu_\nu)_{\mu,\nu\in O_j}\otimes\overline{J^a(0)}]\ (\tilde{y}_\mu)_{\mu\in O_j} \\
&\quad = \left(\sum_{0\leqslant\eta<\mu}\ \sum_{|\nu|=|\eta|} m^\eta_\nu \overline{J^a(\mu-\eta)}\ \tilde{y}_\nu\right)_{\mu\in O_j},
\end{aligned}
$$

for $j=0, 1, ..., k-1$. When $j=0$, we have $m\tilde{y}_0=\overline{J^a(0)}\ \tilde{y}_0$. Since (1.2) holds true, such vector $\tilde{y}_0$ is unique up to a scalar multiplier constant. When $j=1, ..., k-1$, it is easy to see that all the eigenvalues of $(m^\mu_\nu)_{\mu,\nu\in O_j}$ are $\sigma^{-\mu}$, $\mu\in O_j$ and therefore are less than 1 in modulus where $\sigma=(\sigma_1, ..., \sigma_s)^T$ and $\sigma_j$ are all the eigenvalues of $M$. Since $a$ satisfies the condition (1.2), we conclude that the matrices $mI_{r(\#O_j)} - (m^\mu_\nu)_{\mu,\nu\in O_j}\otimes\overline{J^a(0)}$ are invertible for $j>0$ which completes the proof. ∎

When $r=1$, Theorem 3.1 can be reduced to the following form.

COROLLARY 3.2. *Let $a$ and $\tilde{a}$ in $\ell_0(\mathbb{Z}^s)$ be two masks satisfying the discrete biorthogonal relation* (1.9) *with a dilation matrix $M$. Suppose that $\sum_{\beta\in\mathbb{Z}^s} a(\beta)=m:=|\det M|$ and $\tilde{a}$ satisfies the sum rules of order $k$. Let $h^a(\mu):=\sum_{\beta\in\mathbb{Z}^s}\tilde{a}(\beta)\,\beta^\mu/m$ for $\mu\in\mathbb{Z}^s_+$. Then $h^a(\mu)$, $|\mu|<k$ are given by the following recursive relation: for $\mu\in\mathbb{Z}^s_+$ and $|\mu|<k$,*

$$
h^a(\mu)=\delta(\mu)-m^{-1}\sum_{0\leqslant\nu<\mu}(-1)^{|\mu-\nu|}\frac{\mu!}{\nu!(\mu-\nu)!}\,h^a(\nu)\sum_{\beta\in\mathbb{Z}^s}\overline{a(\beta)}\,\beta^{\mu-\nu}.
$$

In the following we shall study how to construct dual masks with arbitrary order of sum rules. For simplicity of exposition, we shall first investigate the case when the primal masks are interpolatory masks. This restriction is not essential and can be removed as demonstrated in [5]. We shall outline such construction for the general case at the end of this section.

CBC ALGORITHM (Coset by Coset Algorithm). Let $a \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ be a mask such that $J^a(0)$ satisfies (1.2) with a dilation matrix $M$, $a(0)$ is invertible and $a(M\beta)=0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$. Let $k$ be a positive integer.

(1) Compute the vectors $\tilde{y}_\mu$, $\mu \in \mathbb{Z}^s_+$ with $|\mu|<k$ as in Theorem 3.1 such that $\tilde{y}_0 \neq 0$;

(2) For any $\varepsilon \in \Omega_M \backslash \{0\}$, choose an appropriate subset $E_\varepsilon$ of $\mathbb{Z}^s$ such that after setting $\tilde{a}(\varepsilon+M\beta)=0$ for all $\beta \in \mathbb{Z}^s \backslash E_\varepsilon$, the following linear system

$$\sum_{0\leqslant \nu \leqslant \mu} (-1)^{|\nu|} J^{\tilde{a}}_\varepsilon(\nu)^T \tilde{y}_{\mu-\nu} = \sum_{|\nu|=|\mu|} m^\mu_\nu \tilde{y}_\nu \qquad \forall \mu \in \mathbb{Z}^s_+, \quad |\mu|<k \tag{3.3}$$

has at least one solution for $\{\tilde{a}(\varepsilon+M\beta)\colon \beta \in E_\varepsilon\}$ where $J^{\tilde{a}}_\varepsilon(\nu)$ is defined in (1.4) as

$$J^{\tilde{a}}_\varepsilon(\nu) := \sum_{\beta \in E_\varepsilon} \tilde{a}(\varepsilon+M\beta)(M^{-1}\varepsilon+\beta)^\nu/\nu!;$$

(3) Construct the coset of $\tilde{a}$ at 0 as follows: for any $\alpha \in \mathbb{Z}^s$,

$$\tilde{a}(M\alpha) = \left[ m\delta(\alpha)\, I_r - \sum_{\varepsilon \in \Omega_M \backslash \{0\}} \sum_{\beta \in E_\varepsilon} \tilde{a}(\varepsilon+M\beta)\, \overline{a(\varepsilon+M(\beta-\alpha))}^T \right] (\overline{a(0)}^T)^{-1}.$$

Then the mask $\tilde{a}$ is a dual mask of the given mask $a$ and $\tilde{a}$ satisfies the sum rules of order $k$.

*Proof.* Since $a(M\beta)=0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$, we can rewrite (1.9) as

$$\begin{aligned} m\delta(\alpha)\, I_r &= \sum_{\varepsilon \in \Omega_M} \sum_{\beta \in \mathbb{Z}^s} \overline{a(\varepsilon+M\beta)}\, \tilde{a}(\varepsilon+M\beta+M\alpha)^T \\ &= \overline{a(0)}\, \tilde{a}(M\alpha)^T + \sum_{\varepsilon \in \Omega_M \backslash \{0\}} \sum_{\beta \in \mathbb{Z}^s} \overline{a(\varepsilon+M\beta-M\alpha)}\, \tilde{a}(\varepsilon+M\beta)^T. \end{aligned}$$

Therefore, under the assumption that $a$ is an interpolatory mask, the discrete biorthogonal relation (1.9) is equivalent to the equality in Step (3). To prove that $\tilde{a}$ satisfies the sum rules of order $k$, by the definition of sum rules in (1.6) and Eq. (3.3), it suffices to verify that

$$\sum_{0\leqslant \nu \leqslant \mu} (-1)^{|\nu|} J^{\tilde{a}}_0(\nu)^T \tilde{y}_{\mu-\nu} = \sum_{|\nu|=|\mu|} m^\mu_\nu \tilde{y}_\nu \qquad \forall \mu \in \mathbb{Z}^s_+, \quad |\mu|<k. \tag{3.4}$$

As in Theorem 3.1, from the discrete biorthogonal relation (1.9), we have

$$m\tilde{y}_\mu = \sum_{\varepsilon \in \Omega_M} \sum_{0 \leqslant \eta \leqslant \mu} \overline{J^a_\varepsilon(\eta)} \sum_{0 \leqslant \nu \leqslant \mu - \eta} (-1)^{|\nu|} J^{\tilde{a}}_\varepsilon(\nu)^T \tilde{y}_{\mu-\eta-\nu}, \qquad \mu \in \mathbb{Z}^s_+.$$

As we know from the proof of Theorem 3.1, $\{\tilde{y}_\mu : |\mu| < k\}$ satisfies the equation

$$m\tilde{y}_\mu = \sum_{\varepsilon \in \Omega_M} \sum_{0 \leqslant \eta \leqslant \mu} \overline{J^a_\varepsilon(\eta)} \sum_{|\nu| = |\mu-\eta|} m^{\mu-\eta}_\nu \tilde{y}_\nu, \qquad |\mu| < k.$$

Therefore, subtracting the last equality from the previous one and using Eq. (3.3), we end up with

$$\sum_{0 \leqslant \eta \leqslant \mu} \overline{J^a_0(\eta)} \left[ \sum_{0 \leqslant \nu \leqslant \mu-\eta} (-1)^{|\nu|} J^{\tilde{a}}_0(\nu)^T \tilde{y}_{\mu-\eta-\nu} - \sum_{|\nu| = |\mu-\eta|} m^{\mu-\eta}_\nu \tilde{y}_\nu \right] = 0$$
$$\forall\ |\mu| < k.$$

Since $J^a_0(0) = a(0)$ and therefore, is invertible, by induction, the above equality implies (3.4) which completes the proof. ∎

The advantage of the above algorithm lies in that we only need to deal with each coset $\varepsilon \in \Omega_M \backslash \{0\}$ separately in Step (2). If $\tilde{a}$ is a dual mask of $a$ such that $\tilde{a}$ satisfies the sum rules of order $k$, then $\tilde{a}$ must be obtained from the above CBC algorithm by choosing some subset $E_\varepsilon$ and a solution in (3.3). The reader may wonder how to choose the subset $E_\varepsilon$ of $\mathbb{Z}^s$ such that there is at least one solution to (3.3). In general, when $E_\varepsilon$ is large enough, there must be a solution to (3.3). In the scalar case $r=1$, this fact was discussed in detail in [5, 17] and we shall sketch the ideas later in this section. For the multiwavelet case $r>1$, let us demonstrate this fact for the univariate case. First, without loss of generality, we may assume that $\tilde{y}_0 = [1, 0, \ldots, 0]^T$. For any $\varepsilon \in \Omega_M \backslash \{0\}$, let $E_\varepsilon$ be any subset of $\mathbb{Z}$ such that $\# E_\varepsilon = k$. Now set $\tilde{a}(\varepsilon + M\beta) = 0$ for any $\beta \in \mathbb{Z} \backslash E_\varepsilon$. For any matrix $A$, let $A[i, j]$ denote its $(i, j)$ entry. Set $\tilde{a}(\varepsilon + M\beta)[i, j] = 0$ (or any number as you want) for all $\beta \in E_\varepsilon$, $i = 2, \ldots, r$ and $j = 1, \ldots, r$. Then (3.3) has a unique solution for $\{\tilde{a}(\varepsilon + M\beta)[1, j] : \beta \in E_\varepsilon, j = 1, \ldots, r\}$ since in this case (3.3) is reduced to the equation

$$\sum_{\beta \in E_\varepsilon} \tilde{a}(\varepsilon + M\beta)[1, j](M^{-1}\varepsilon + \beta)^\mu = g_{\varepsilon, j}(\mu), \qquad 0 \leqslant \mu < k,$$

where $g_{\varepsilon, j}(\mu)$ are constants derived from (3.3). It is evident that the above equation has a unique solution since its coefficient matrix is a Vandermonde matrix.

From the above CBC algorithm, it is not difficult to see that for any positive integer $k$, there always exists a dual mask which satisfies the sum rules of order $k$. We shall prove this fact for the general case at the end of this section. In the case $r=1$, Eq. (3.3) becomes

$$\sum_{\beta\in\mathbb{Z}^s} \tilde{a}(\varepsilon+M\beta)(\varepsilon+M\beta)^{\mu}=h^a(\mu) \qquad \forall\ |\mu|<k, \quad \mu\in\mathbb{Z}^s_+,$$

where $h^a(\mu)$ are calculated from Corollary 3.2. See [17] for more details. When $r=1$, in [5], we demonstrated that given any primal mask $a$, the CBC algorithm in [5, 17] reduces the construction of all the dual masks of $a$ such that the dual masks satisfy the sum rules of order $k$ to a problem of obtaining a sequence $b$ in $\ell_0(\mathbb{Z}^s)$ such that $b(0)=1$, $b(M\beta)=0$ for all $\beta\in\mathbb{Z}^s\backslash\{0\}$, and for each $\varepsilon\in\Omega_M\backslash\{0\}$,

$$\sum_{\beta\in\mathbb{Z}^s} \tilde{b}(\varepsilon+M\beta)(\varepsilon+M\beta)^{\mu}=g_\varepsilon(\mu) \qquad \forall\ |\mu|<k, \quad \mu\in\mathbb{Z}^s_+,$$

where $g_\varepsilon(\mu)$, $|\mu|<k$ can be similarly computed as $h^a(\mu)$.

In the scalar case $r=1$, an algorithm is proposed in [5, 17, 20] to concretely implement the general CBC algorithm. Let $a$ be a given scalar primal mask which is symmetric about the origin. For any positive integer $k$, such algorithm gives a unique dual mask of $a$ such that the dual mask satisfies the sum rules of order $2k$ and is symmetric about the origin. See [5, 17, 20] for some examples of scalar biorthogonal wavelets constructed by such CBC algorithm.

In the following, we shall study how to construct dual masks in the most general case. We shall demonstrate in a constructive way that for any given primal mask $a$ with a dilation matrix $M$, there always exists an dual mask $\tilde{a}$ of $a$ such that $\tilde{a}$ satisfies the sum rules of any preassigned order.

For any mask $a\in(\ell_0(\mathbb{Z}^s))^{r\times r}$ and for any $\varepsilon\in\Omega_M$, by $a_\varepsilon$ we denote the sequence

$$a_\varepsilon(\beta):=a(\varepsilon+M\beta), \qquad \beta\in\mathbb{Z}^s.$$

Under the lexicographic order, $\Omega_M$ is an ordered set, say $\Omega_M=\{\varepsilon_1, ..., \varepsilon_m\}$. It is a well known fact that $a$ is a primal mask with a dilation matrix $M$ if and only if

$$\operatorname{Rank}(\mathbf{a}_{\varepsilon_1}(z), ..., \mathbf{a}_{\varepsilon_m}(z))=r \qquad \forall z\in\mathbb{T}^s:=\{(z_1, ..., z_s)\colon |z_1|=\cdots=|z_s|=1\},$$

where

$$\mathbf{a}_\varepsilon(z):=\sum_{\beta\in\mathbb{Z}^s} \mathbf{a}(\varepsilon+M\beta)\, Z^{\beta}, \qquad z\in\mathbb{T}^s.$$

Moreover, $\tilde{a}$ is a dual mask of $a$ if and only if

$$\sum_{\varepsilon\in\Omega_M} \overline{\mathbf{a}_\varepsilon(z)}\, \tilde{\mathbf{a}}_\varepsilon(z)^T=mI_r \qquad \forall z\in\mathbb{T}^s.$$

The following known fact is employed in the lifting scheme in some sense. For detail about lifting schemes, see [32, 41] and references therein.

PROPOSITION 3.3. *Let $a$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ be a primal mask with a dilation matrix $M$. Suppose there exist sequences $\tilde{a}^\omega$, $\omega\in\Omega_M$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ such that*

$$\sum_{\varepsilon\in\Omega_M} \overline{\mathbf{a}_\varepsilon(z)}\, \tilde{\mathbf{a}}_\varepsilon^\omega(z)^T = m\delta(\omega)\, I_r \qquad \forall\omega\in\Omega_M$$

*and the matrix $(\tilde{\mathbf{a}}_\varepsilon^\omega(z))_{\omega,\varepsilon\in\Omega_M}$ is invertible for all $z\in\mathbb{T}^s$. Then for any sequence $\tilde{a}$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$, $\tilde{a}$ is a dual mask of $a$ if and only if there exists a sequence $b$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ with $\mathbf{b}_0(z)=I_r$ such that*

$$\tilde{\mathbf{a}}_\omega(z) = \sum_{\varepsilon\in\Omega_M} \mathbf{b}_\varepsilon(z)\, \tilde{\mathbf{a}}_\omega^\varepsilon(z) \qquad \forall\omega\in\Omega_M, \quad z\in\mathbb{T}^s. \tag{3.5}$$

*Proof.* If $\tilde{a}$ satisfies (3.5), it is easy to verify that

$$\begin{aligned}\sum_{\omega\in\Omega_M} \overline{\mathbf{a}_\omega(z)}\, \tilde{\mathbf{a}}_\omega(z)^T &= \sum_{\varepsilon\in\Omega_M}\sum_{\omega\in\Omega_M} \overline{\mathbf{a}_\omega(z)}\, \tilde{\mathbf{a}}_\omega^\varepsilon(z)^T\, \mathbf{b}_\varepsilon(z)^T\\ &= m\,\mathbf{b}_0(z)^T = mI_r.\end{aligned}$$

Thus, $\tilde{a}$ is a dual mask of $a$.

Conversely, if $\tilde{a}$ is a dual mask of $a$, setting

$$(\mathbf{b}_{\varepsilon_1}(z), ..., \mathbf{b}_{\varepsilon_{\mathbf{m}}}(z)) := (\tilde{\mathbf{a}}_{\varepsilon_1}(z), ..., \tilde{\mathbf{a}}_{\varepsilon_{\mathbf{m}}}(z))(\tilde{\mathbf{a}}_\varepsilon^\omega(z))^{-1}_{\omega,\varepsilon\in\Omega_M},$$

where $\{\varepsilon_1, ..., \varepsilon_m\} = \Omega_M$, then $\mathbf{b}_0(z)=I_r$ and (3.5) holds true. ∎

It is well known that the invertible matrix $(\tilde{\mathbf{a}}_\varepsilon^\omega(z))_{\omega,\varepsilon\in\Omega_M}$ is used in deriving the associated dual wavelet masks from $a$ and $\tilde{a}$ and such matrix can be constructed from the mask $a$ by an algorithm proposed in [33]. With the help of Theorem 3.1 and Proposition 3.3, we demonstrate the following (obvious?) fact:

THEOREM 3.4. *Let $a$ be a primal mask in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ with a dilation matrix $M$. For any positive integer $k$, there exists a dual mask $\tilde{a}$ of $a$ such that $\tilde{a}$ satisfies the sum rules of order $k$*

*Proof.* Since $a$ is a primal mask, by Quillen–Suslin Theorem, there exist sequences $\tilde{a}^\omega$ $(\omega\in\Omega_M)$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ such that the conditions in Proposition 3.3 are satisfied. Such sequences $\tilde{a}^\omega$ can be constructed from $a$ by an algorithm proposed in [33]. Let $\tilde{a}$ be a sequence in $(\ell_0(\mathbb{Z}^s))^{r\times r}$ given by

$$\tilde{\mathbf{a}}_\omega(z) := \sum_{\varepsilon\in\Omega_M} \mathbf{b}_\varepsilon(z)\, \tilde{\mathbf{a}}_\omega^\varepsilon(z), \qquad \omega\in\Omega_M, \tag{3.6}$$

where $b \in (\ell_0(\mathbb{Z}^s))^{r \times r}$ is a sequence to be determined. In order to obtain a dual mask $\tilde{a}$ (in the above form) of $a$ such that $\tilde{a}$ satisfies the sum rules of order $k$, by Proposition 3.3, it suffices to demonstrate that there does exist a sequence $b$ such that $\mathbf{b}_0(z) = I_r$ and $\tilde{a}$ satisfies the sum rule equations

$$\sum_{0 \leqslant \nu \leqslant \mu} (-1)^{|\nu|} J_\omega^{\tilde{a}}(\nu)^T \tilde{y}_{\mu-\nu} = \sum_{|\nu|=|\mu|} m_\nu^\mu \tilde{y}_\nu \qquad \forall \mu \in \mathbb{Z}_+^s, \quad |\mu| < k, \quad \omega \in \Omega_M, \tag{3.7}$$

where the vectors $\tilde{y}_\mu$, $|\mu| < k$ are computed from Theorem 3.1 with $\tilde{y}_0 \neq 0$. Without loss of generality, we may assume $\tilde{y}_0 = [1, 0, ..., 0]^T$. From (3.6), for any $\nu \in \mathbb{Z}_+^s$ and $\omega \in \Omega_M$, we deduce that

$$\begin{aligned} J_\omega^{\tilde{a}}(\nu) &= \sum_{\alpha \in \mathbb{Z}^s} \tilde{a}_\omega(\alpha) \frac{(M^{-1}\omega + \alpha)^\nu}{\nu!} \\ &= \sum_{\varepsilon \in \Omega_M} \sum_{\alpha \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} b_\varepsilon(\beta) \frac{(M^{-1}\varepsilon + \beta)^\eta}{\eta!} \\ &\quad \times \sum_{\alpha \in \mathbb{Z}^s} \tilde{a}_\omega^\varepsilon(\alpha - \beta) \frac{(M^{-1}(\omega - \varepsilon) + \alpha - \beta)^{\nu - \eta}}{(\nu - \eta)!} \\ &= \sum_{0 \leqslant \eta \leqslant \nu} \sum_{\varepsilon \in \Omega_M} J_\varepsilon^b(\eta) F_\omega^\varepsilon(\nu - \eta), \end{aligned}$$

where $F_\omega^\varepsilon(\mu) := \sum_{\beta \in \mathbb{Z}^s} \tilde{a}_\omega^\varepsilon(\beta)(M^{-1}(\omega - \varepsilon) + \beta)^\mu / \mu!$ for any $\mu \in \mathbb{Z}_+^s$. Thus, the linear system (3.7) can be rewritten: for all $|\mu| < k$ and $\omega \in \Omega_M$,

$$\sum_{0 \leqslant \nu \leqslant \mu} \sum_{0 \leqslant \eta \leqslant \nu} \sum_{\varepsilon \in \Omega_M} (-1)^{|\nu|} F_\omega^\varepsilon(\nu - \eta)^T J_\varepsilon^b(\eta)^T \tilde{y}_{\mu - \nu} = \sum_{|\nu| = |\mu|} m_\nu^\mu \tilde{y}_\nu. \tag{3.8}$$

Therefore, after changing the order of summation in the left side of (3.8), we get

$$\sum_{0 \leqslant \eta \leqslant \mu} \sum_{0 \leqslant \nu \leqslant \mu - \eta} \sum_{\varepsilon \in \Omega_M} (-1)^{|\nu + \eta|} F_\omega^\varepsilon(\nu)^T J_\varepsilon^b(\eta)^T \tilde{y}_{\mu - \eta - \nu} = \sum_{|\nu| = |\mu|} m_\nu^\mu \tilde{y}_\nu.$$

In other words, (3.7) can be rewritten in the form

$$\begin{aligned} &\sum_{\varepsilon \in \Omega_M} F_\omega^\varepsilon(0)^T (-1)^{|\mu|} J_\varepsilon^b(\mu)^T \tilde{y}_0 \\ &\qquad + \sum_{0 \leqslant \eta \leqslant \mu} \sum_{0 \leqslant \nu \leqslant \mu - \eta} \sum_{\varepsilon \in \Omega_M} (-1)^{|\nu + \eta|} F_\omega^\varepsilon(\nu)^T J_\varepsilon^b(\eta)^T \tilde{y}_{\mu - \eta - \nu} \\ &\quad = \sum_{|\nu| = |\mu|} m_\nu^\mu \tilde{y}_\nu, \qquad \forall \mu \in \mathbb{Z}_+^s, \quad |\mu| < k, \quad \omega \in \Omega_M \end{aligned}$$

with a sequence $b$ to be determined. Note that each $b_\varepsilon(\beta)$ is an $r\times r$ matrix. By $b_\varepsilon(\beta)[i, j]$ we denote the $(i, j)$ entry of the matrix $b_\varepsilon(\beta)$. Regard all $b_\varepsilon(\beta)[i, j]$ as parameters for all $\varepsilon\in\Omega_M$, $\beta\in\mathbb{Z}^s$, $2\leqslant i\leqslant r$ and $1\leqslant j\leqslant r$. Let $c$ in $(\ell_0(\mathbb{Z}^s))^r$ be given by $c(\beta)=(b(\beta)[1,1], ..., b(\beta)[1,r])^T$. Then it is easily seen that $J^c_\omega(\mu)=J^b_\omega(\mu)^T\,\tilde{y}_0$ since $\tilde{y}_0=[1,0,...,0]^T$. Therefore, we end up with the equations

$$\sum_{\varepsilon\in\Omega_M} F^\varepsilon_\omega(0)^T\,(-1)^{|\mu|}\,J^c_\varepsilon(\mu)$$
$$+\sum_{0\leqslant\nu<\mu}\sum_{\varepsilon\in\Omega_M} G^\varepsilon_{\omega,\nu}J^c_\varepsilon(\nu)=g^\omega_\mu,\qquad \forall\omega\in\Omega_M,\quad |\mu|<k,\tag{3.9}$$

where $G^\varepsilon_{\omega,\nu}$ are constant matrices, and $g^\omega_\mu$ are vectors with parameters $\{b_\varepsilon(\beta)[i,j]\colon \varepsilon\in\Omega_M,\ \beta\in\mathbb{Z}^s,\ 2\leqslant i\leqslant r,\ 1\leqslant j\leqslant r\}$. Since $F^\varepsilon_\omega(0)=\tilde{\mathbf{a}}^\varepsilon_\omega(1)$ and $(\tilde{\mathbf{a}}^\varepsilon_\omega(z))_{\varepsilon,\omega\in\Omega_M}$ is invertible, the matrix $(F^\varepsilon_\omega(0))_{\varepsilon,\omega\in\Omega_M}$ has full rank which implies that the equations (3.9) can be uniquely solved as

$$J^c_\varepsilon(\mu)=f^\varepsilon_\mu\qquad \forall\varepsilon\in\Omega_M,\quad \mu\in\mathbb{Z}^s_+,\quad |\mu|<k,\tag{3.10}$$

where $f^\varepsilon_\mu$ are vectors with parameters $\{b_\varepsilon(\beta)[i,j]\colon \varepsilon\in\Omega_M,\ \beta\in\mathbb{Z}^s,\ 2\leqslant i\leqslant r,\ 1\leqslant j\leqslant r\}$.

We now prove that the following linear system

$$J^c_0(\mu)=f^0_\mu\qquad \forall\mu\in\mathbb{Z}^s_+,\quad |\mu|<k\tag{3.11}$$

has a solution. Note that (3.6) and the biorthogonal condition imply that

$$m\,\mathbf{b}_0(z)=\sum_{\varepsilon\in\Omega_M}\overline{\mathbf{a}_\varepsilon(z)}\,\tilde{\mathbf{a}}_\varepsilon(z)^T\qquad \forall z\in\mathbb{T}^s$$

which can be rewritten as

$$mb(M\alpha)=\sum_{\varepsilon\in\Omega_M}\sum_{\beta\in\mathbb{Z}^s}\overline{a(\varepsilon+M\beta)}\,\tilde{a}_\varepsilon(\varepsilon+M\beta+M\alpha)^T,\qquad \alpha\in\mathbb{Z}^s.\tag{3.12}$$

From (3.12), by the same argument as in Theorem 3.1, we have

$$J^b_0(\nu)=\sum_{\varepsilon\in\Omega_M}\sum_{0\leqslant\eta\leqslant\nu}(-1)^{|\eta|}\,\overline{J^a_\varepsilon(\eta)}\,J^{\tilde{a}}_\varepsilon(\nu-\eta)^T,\qquad \nu\in\mathbb{Z}^s_+$$

which is similar to (3.1). Since $\tilde{\mathbf{a}}$ must satisfy (3.7), by a similar argument as in Theorem 3.1, we have

$$m\sum_{0\leqslant\nu\leqslant\mu}(-1)^{|\nu|}\,J^b_0(\nu)\,\tilde{y}_{\mu-\nu}=\sum_{0\leqslant\eta\leqslant\mu}\sum_{|\nu|=|\eta|}m^\eta_\nu\overline{J^a(\mu-\eta)}\,\tilde{y}_\nu\qquad \forall\ |\mu|<k.\tag{3.13}$$

Since $\tilde{y}_\mu$, $|\mu|<k$ satisfy (3.2), we deduce that

$$\sum_{0\leqslant \nu\leqslant \mu} (-1)^{|\nu|} J_0^b(\nu)\, \tilde{y}_{\mu-\nu}=\tilde{y}_\mu \qquad \forall \mu\in \mathbb{Z}_+^s, \quad |\mu|<k. \tag{3.14}$$

By the uniqueness of $f_\mu^\varepsilon$, $\varepsilon\in\Omega_M$ and $|\mu|<k$, we conclude that the linear system (3.11) must be equivalent to the linear system (3.14) since $J_0^c(\mu)$, $|\mu|<k$ can also be uniquely solved from (3.14).

It is evident that $\mathbf{b}_0(z)=I_r$ is a solution to (3.14). Therefore, $\mathbf{b}_0(z)=I_r$ is a solution to (3.11). Set $b_0(M\beta)=\delta(\beta)\, I_r$ and $b_\varepsilon(\beta)[i, j] :=0$ (or any other numbers as long as $b_\varepsilon$ in $(\ell_0(\mathbb{Z}^s))^{r\times r}$) for all $\varepsilon\in\Omega_M\backslash\{0\}$, $\beta\in\mathbb{Z}^s$, $2\leqslant i\leqslant r$ and $1\leqslant j\leqslant r$. Then for each $\varepsilon\in\Omega_M\backslash\{0\}$, $f_\mu^\varepsilon$ in (3.10) are constants.

To demonstrate that for each $\varepsilon\in\Omega_M\backslash\{0\}$, (3.10) has at least one solution, it suffices to prove that for each $\varepsilon\in\Omega_M\backslash\{0\}$ and $j=1, ..., r$, the following linear system

$$\sum_{\beta\in E_\varepsilon} C(\beta)(M^{-1}\varepsilon+\beta)^\mu/\mu\,! = h_\mu^\varepsilon \qquad \forall \mu\in\mathbb{Z}_+^s, \quad |\mu|<k \tag{3.15}$$

has a solution for $C(\beta)$, $\beta\in E_\varepsilon$ where $E_\varepsilon$ is a subset of $\mathbb{Z}^s$, $C(\beta)=b_\varepsilon(\beta)[1, j]$ and $h_\mu^\varepsilon :=f_\mu^\varepsilon[j]$. It is evident that (3.15) can be rewritten as

$$\sum_{\beta\in E_\varepsilon} C(\beta)\, \beta^\mu = \tilde{h}_\mu^\varepsilon \qquad \forall \mu\in\mathbb{Z}_+^s, \quad |\mu|<k \tag{3.16}$$

for some constants $\tilde{h}_\mu^\varepsilon$. The above linear system always has a solution for $\{C(\beta)\colon \beta\in E_\varepsilon\}$ when $E_\varepsilon$ is large enough. For example, if $E_\varepsilon :=\{\beta\in\mathbb{Z}_+^s : |\beta|<k\}$, then (3.16) has a unique solution for $\{C(\beta)\colon \beta\in E_\varepsilon\}$. For the solvability of (3.16), the reader is referred to [35] on multivariate polynomial interpolation. ∎

The proof of Theorem 3.4 illustrates the general procedure to construct all dual masks of any given primal mask such that the dual masks satisfy the sum rules of any preassigned order. Though a general CBC algorithm for the general primal mask is possible but the exposition of such algorithm is much more complicated and technical. For the sake of simplicity, we shall not discuss the details here. In general, there are three major steps in constructing a dual mask with arbitrary order of sum rules. First, solve the linear system given in the discrete biorthogonal relation (1.9) by Proposition 3.3 or by any other method. Second, for any fixed positive integer $k$, compute the vectors $\tilde{y}_\mu$, $|\mu|<k$ as in Theorem 3.1. Finally, solve the linear system given in the sum rule equations (1.6). The existence of a solution to (1.6) is guaranteed by Theorem 3.4 by appropriately choosing the support of the dual masks. The general CBC algorithm for any primal mask with $r=1$ is explicitly given in [5]. Therefore, for any primal mask $a$, we can

employ the CBC algorithm to obtain a dual mask $\tilde{a}$ of $a$ such that $\tilde{a}$ satisfies the sum rules of any preassigned order. If the subdivision schemes associated with the primal mask $a$ and the dual mask $\tilde{a}$ converge in the $L_2$ norm, respectively, then we have the primal function vector $\phi^a$ and its dual function vector $\phi^{\tilde{a}}$ such that the biorthogonal relation (1.8) holds. It is easy to numerically check the $L_2$ convergence of a subdivision scheme [11, 18, 30, 31, 39].

Now an important question which we didn't answer yet is the following: how to obtain primal masks? In Section 4, we shall propose a general way of constructing Hermite interpolatory masks and such Hermite interpolatory masks consist of an important family of primal masks.

## 4. HERMITE INTERPOLATORY MASKS AND BIORTHOGONAL MULTIWAVELETS

In this section, we shall study an important family of primal masks—Hermite interpolatory masks. Hermite interpolatory masks are employed and are useful in curve design in computer aided geometric design [13]. Several examples of biorthogonal multiwavelets are presented to illustrate the general theory developed in Section 3. For simplicity, all the examples are given for the case $s=1$, $r=2$, and $M=(2)$ though the CBC algorithm developed in Section 3 can be easily applied to the general case.

Let $\phi=(\phi_1, ..., \phi_r)^T$ be a function vector. We say that $\phi$ is a *Hermite interpolant* if all the functions $\phi_j$, $j=1, ..., r$ belong to $C^{r-1}(\mathbb{R})$ and satisfy

$$\phi_j^{(l)}(k)=\delta(j-l-1)\,\delta(k) \qquad \forall k\in\mathbb{Z}, \quad l=0, ..., r-1, \quad j=1, ..., r, \qquad (4.1)$$

where $\phi_j^{(l)}$ means the $l$th derivative of $\phi_j$. Let $\phi=(\phi_1, ..., \phi_r)^T$ and $c_k^l$, $k\in\mathbb{Z}$, $l=0, ..., r-1$ be given data. We can construct a function $f$ as

$$f(x)=\sum_{l=0}^{r-1}\sum_{k\in\mathbb{Z}} c_k^l\phi_{l+1}(x-k), \qquad x\in\mathbb{R}.$$

If $\phi$ is a Hermite interpolant, then $f^{(l)}(k)=c_k^l$ for all $k\in\mathbb{Z}$ and $l=0, ..., r-1$.

LEMMA 4.1. *Let $a\in(\ell_0(\mathbb{Z}))^{r\times r}$ be a finitely supported mask and $\phi$ be the normalized solution of the refinement equation* (1.1) *with mask a and the dilation factor $M=(d)$ where d is an integer. If $\phi$ is a Hermite interpolant, then $a(dj)=\delta(j)\,\mathrm{diag}(1, d^{-1}, ..., d^{1-r})$ for all $j\in\mathbb{Z}$ and a satisfies the sum rules of order r as defined in* (1.6) *with $\{y_0, ..., y_{r-1}\}$ where $y_j=e_{j+1}$, $j=0, ..., r-1$ and $e_j$ is the jth coordinate unit vector in $\mathbb{R}^r$.*

*Proof.* From the refinement equation (1.1), we have

$$[\phi(k), \phi'(k), ..., \phi^{(r-1)}(k)] = \sum_{j\in\mathbb{Z}} a(j)[\phi(dk-j), d\phi'(dk-j), ..., d^{r-1}\phi^{(r-1)}(dk-j)]$$

for all $k\in\mathbb{Z}$. By (4.1), it is evident that $a(dj)=\delta(j)\,\mathrm{diag}(1, d^{-1}, ..., d^{1-r})$ for all $j\in\mathbb{Z}$. Note that the shifts of $\phi$ are linearly independent since $\phi$ is a Hermite interpolant. Therefore, $\phi\in C^{r-1}$ implies that $\phi$ has accuracy order $r$. By Theorem 2.4, $a$ must satisfy the sum rules of order $r$ with some vectors $\{y_0, y_1, ..., y_{r-1}\}$ such that

$$\frac{x^{r-1}}{(r-1)!} = \sum_{0\leqslant\nu\leqslant r-1}\sum_{j\in\mathbb{Z}} \frac{j^\nu}{\nu!}\, y^T_{r-1-\nu}\phi(x-j) \qquad \forall x\in\mathbb{R}.$$

Therefore,

$$\left[\frac{x^{r-1}}{(r-1)!}, ..., x, 1\right] = \sum_{0\leqslant\nu\leqslant r-1}\sum_{j\in\mathbb{Z}} \frac{j^\nu}{\nu!}\, y^T_{r-1-\nu}[\phi(x-j), ..., \phi^{(r-1)}(x-j)], \qquad x\in\mathbb{R}.$$

Since $\phi$ is a Hermite interpolant, from (4.1), we have

$$\left[\frac{k^{r-1}}{(r-1)!}, \frac{k^{r-2}}{(r-2)!}, ..., k, 1\right] = \sum_{0\leqslant\nu\leqslant r-1} \frac{k^\nu}{\nu!}\, y^T_{r-1-\nu} \qquad \forall k\in\mathbb{Z},$$

which implies $y_j=e_{j+1}$ for all $j=0, ..., r-1$. ∎

For any $a\in(\ell_0(\mathbb{Z}))^{r\times r}$, we say that $a$ is a *Hermite interpolatory mask* with the dilation factor $M=(d)$ if $a(dj)=\delta(j)\,\mathrm{diag}(1, d^{-1}, ..., d^{1-r})$ for all $j\in\mathbb{Z}$ and $a$ satisfies the sum rules of order $r$ as defined in (1.6) with $\{y_0, y_1, ..., y_{r-1}\}$ where $y_j=e_{j+1}$, $j=0, ..., r-1$ and $e_j$ is the $j$th coordinate unit vector in $\mathbb{R}^r$. Let $\phi^a$ be the normalized solution of (1.1) with a mask $a$ and a dilation factor $M=(d)$. If $\phi^a$ is a Hermite interpolant, by Lemma 4.1, it is necessary that $a$ is a Hermite interpolatory mask with the dilation factor $M=(d)$.

The following theorem gives us a family of Hermite interpolatory masks. For simplicity of proof, we only deal with $s=1$, $M=(2)$ and $r=2$ in the following result.

THEOREM 4.2. *For any $N\in\mathbb{N}$, there is a unique mask $b_N$ in $(\ell_0(\mathbb{Z}))^{2\times 2}$ with the dilation factor $M=(2)$ such that*

(a) *$b_N$ is supported on* $[1-2N, 2N-1]$;

(b) *$b_N$ is a Hermite interpolatory mask*;

(c) *$b_N$ satisfies the sum rules of order* $4N$.

*Proof.* By the definition of sum rules, it is equivalent to proving that there is a unique mask $b_N$ in $(\ell_0(\mathbb{Z}))^{2\times 2}$ such that $b_N$ is supported on $[1-2N, 2N-1]$, $b_N(2j)=\delta(j)\operatorname{diag}(1, 1/2)$ and

$$\sum_{0\leqslant\nu\leqslant\mu} (-1)^{|\nu|}\, J_1^{b_N}(\nu)^T\, y_{\mu-\nu}=2^{-|\mu|}y_\mu \qquad \forall 0\leqslant\mu<4N, \tag{4.2}$$

where $y_0^T=[1,0]$, $y_1^T=[0,1]$ and $y_\nu=0$ for $2\leqslant\nu<4N$. Recall that $b_N(1+2j)[i,k]$ denotes the $(i,k)$ entry of the matrix $b_N(1+2j)$. It suffices to prove that the following linear systems for all $0\leqslant\mu<4N$,

$$\sum_{j=-N}^{N-1} b_N(1+2j)[1,1]\,\frac{(1/2+j)^\mu}{\mu!} - \sum_{j=-N}^{N-1} b_N(1+2j)[2,1]\,\frac{(1/2+j)^{\mu-1}}{(\mu-1)!}=\delta(\mu) \tag{4.3}$$

and

$$\sum_{j=-N}^{N-1} b_N(1+2j)[1,2]\,\frac{(1/2+j)^\mu}{\mu!} - \sum_{j=-N}^{N-1} b_N(1+2j)[2,2]\,\frac{(1/2+j)^{\mu-1}}{(\mu-1)!}=-\frac{\delta(\mu-1)}{2} \tag{4.4}$$

have a unique solution. Here, we used the convention $(-1)!=\infty$.

Let $A$ be a $4N$ by $4N$ square matrix given by $A[\mu, 2(j+N)]=(1/2+j)^\mu/\mu!$ and $A[\mu, 2(j+N)+1]=(1/2+j)^{\mu-1}/(\mu-1)!$ for $0\leqslant\mu<4N$ and $j=-N, ..., N-1$. Then the linear system (4.3) can be rewritten in the matrix form as

$$Ax=[1,0,...,0]^T,$$

where $x[2(j+N)]=b_N(1+2j)[1,1]$ and $x[2(j+N)+1]=-b_N(1+2j)[2,1]$ for $j=-N, ..., N-1$.

To see that the linear system (4.3) has a unique solution, it suffices to demonstrate that $A$ is invertible, i.e., if $\lambda^T A=0$ for some vector $\lambda=(\lambda_\mu)_{0\leqslant\mu<4N}$, then $\lambda=0$. Let

$$F(x):=\sum_{0\leqslant\mu<4N}\lambda_\mu\frac{x^\mu}{\mu!}.$$

Then $\lambda^T A = 0$ is equivalent to

$$F(1/2+j)=0 \qquad \text{and} \qquad F'(1/2+j)=0 \quad \forall j=-N, ..., N-1. \tag{4.5}$$

It is well known that the only solution to (4.5) is $F=0$. Therefore, $\lambda=0$ and (4.3) has a unique solution. Similarly, (4.4) has a unique solution. ∎

By a similar argument as in the proof of Theorem 4.2, we have the following result which is a generalization of [19, Theorem 2.1]. In the following, by $e_j$ we denote the $j$th unit vector in $\mathbb{R}^r$.

THEOREM 4.3. *Let $d$ be an integer such that $d>1$. Then for any positive integers $l$ and $h$, there exists a unique mask $a$ in $(\ell_0(\mathbb{Z}))^{r\times r}$ with the dilation factor $M=(d)$ such that*

(a) *$a$ is supported on $[1-dl, dh-1]$;*

(b) *$a(dj)=\delta(j)\,\mathrm{diag}(1, d^{-1}, d^{-2}, ..., d^{1-r})$ for all $j\in\mathbb{Z}$;*

(c) *$a$ satisfies the sum rules of order $r(l+h)$ with $\{y_j\colon 0\leqslant j<r(l+h)\}$ where $y_j=e_{j+1}$ for $j=0, ..., r-1$ and $y_j=0$ for $r\leqslant j<r(l+h)$.*

Though in the above results we only deal with the Hermite interpolatory masks, we shall analyze the smoothness and other important properties of the normalized solution $\phi^a$ associated with the Hermite interpolatory mask $a$ which is constructed in Theorem 4.3 elsewhere. It is evident that any Hermite interpolatory mask is a primal mask. Before introducing several examples, let us recall some facts about $L_p$ smoothness of refinable function vectors.

For any $1\leqslant p\leqslant\infty$ and $0<\eta\leqslant 1$, the *Lipschitz space $Lip(\eta, L_p(\mathbb{R}^s))$* consists of those functions $f$ in $L_p(\mathbb{R}^s)$ for which

$$\|f-f(\cdot-t)\|_p\leqslant C\,\|t\|^\eta \qquad \forall t\in\mathbb{R}^s,$$

where $C$ is a constant independent of $t$.

The $L_p$ smoothness of a function $f\in L_p(\mathbb{R}^s)$ in the $L_p$ norm is described by its *$L_p$ critical exponent $\nu_p(f)$* defined by

$$\nu_p(f):=\sup\,\{n+\eta : \partial^\alpha f\in Lip(\eta, L_p(\mathbb{R}^s))\ \forall\ |\alpha|=n,\ \alpha\in\mathbb{Z}_+^s\},$$

where $\alpha=(\alpha_1, ..., \alpha_s)$, $|\alpha|:=|\alpha_1|+\cdots+|\alpha_s|$, and

$$\partial^\alpha f:=\frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1}\cdots\partial x_s^{\alpha_s}}.$$

Evidently $\nu_\infty(f)$ is exactly the Hölder exponent of a function as defined in the current literature (see [11]). For any $\nu > 0$, the Sobolev space $W_2^\nu(\mathbb{R}^s)$ contains of all the functions $f \in L_2(\mathbb{R}^s)$ for which

$$\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 \, (1+|\xi|^\nu)^2 \, d\xi < \infty.$$

It is well known that $\nu_2(f) = \sup\{\nu\colon f \in W_2^\nu(\mathbb{R}^s)\}$.

If $f = (f_1, ..., f_r)^T$ is a function vector in $(L_p(\mathbb{R}^s))^r$, then its $L_p$ critical exponent $\nu_p(f)$ is defined by

$$\nu_p(f) := \min\{\nu_p(f_1), ..., \nu_p(f_r)\}.$$

For any sequence $\lambda \in (\ell_0(\mathbb{Z}^s))^{r\times r}$, $\|\lambda\|_p := (\sum_{\alpha \in \mathbb{Z}^s} |\lambda(\alpha)|^p)^{1/p}$ where $|\cdot|$ denotes a matrix norm. The difference operator $\nabla_j$ is defined to be

$$\nabla_j \lambda := \lambda - \lambda(\cdot - e_j), \qquad \lambda \in (\ell_0(\mathbb{Z}^s))^{r\times r}.$$

When $s = 1$ and $m = (2)$, the $L_p$ smoothness of $\phi^a$ was characterized in [30, 34]. For $L_2$ smoothness of refinable function vectors, also see [31, 38, 39]. By using [17, Theorem 3.1], the result about $L_p$ smoothness of refinable functions in [17, 30] can be easily generalized to the general case. By a similar argument as in [17, 30], we have the following result:

THEOREM 4.4. *Let* $\phi = (\phi_1, ..., \phi_r)^T \in (L_p(\mathbb{R}^s))^r$ $(1 \leqslant p \leqslant \infty)$ *be the normalized distributional solution of the refinement equation* (1.1) *with a finitely supported mask a and a dilation matrix M with* $m := |\det M|$ *such that* $M^j$ *is a multiple of the identity matrix for some integer j. For any nonnegative integer k, define*

$$\sigma^M_{k,p}(a) := \max\{\lim_{n\to\infty} \|\nabla_j^k S_a^n \delta_I\|_p^{1/n} \colon j = 1, ..., s\},$$

*where* $\delta_I(\alpha) = \delta(\alpha)\, I_s$ *for* $\alpha \in \mathbb{Z}^s$. *Then* $\nu_p(\phi) \geqslant s/p - s\log_m \sigma^M_{k,p}(a)$. *If the shifts of* $\phi$ *are stable and* $k > \nu_p(\phi)$, *then*

$$\nu_p(\phi) = s/p - s \log_m \sigma^M_{k,p}(a).$$

*Let* $T_a$ *be the linear operator on* $(\ell_0(\mathbb{Z}^s))^{r\times r}$ *defined by*

$$T_a\lambda(\alpha) = \sum_{\beta\in\mathbb{Z}^s} \sum_{\gamma\in\mathbb{Z}^s} a(M\alpha-\beta)\,\lambda(\beta+\gamma)\,\overline{a(\gamma)}^T,$$
$$\alpha\in\mathbb{Z}^s, \quad \lambda \in (\ell_0(\mathbb{Z}^s))^{r\times r},$$

*and let* $\Delta_j$ *denote the operator defined by* $\Delta_j \lambda = -\lambda(\cdot - e_j) + 2\lambda - \lambda(\cdot + e_j)$. *Then*

$$\sigma^M_{k,p}(a) = \sqrt{\rho(T_a|_W)},$$

*where* $\rho(T_a|_W)$ *is the spectral radius of* $T_a$ *restricted to the finite dimensional space* $W$, *and* $W$ *is the minimal* $T_a$*-invariant space generated by* $\{\Delta_j^k \delta_I : j = 1, ..., s\}$.

Throughout rest of this section, by $U$ we denote the following $2 \times 2$ matrix:

$$U := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In the following we shall first present several examples of Hermite interpolants constructed in Theorem 4.2. Then we shall use the CBC algorithm developed in Section 3 to construct several examples of biorthogonal wavelets. Throughout rest of this section, we assume that $s = 1$, $M = (2)$ and $r = 2$.

EXAMPLE 4.5. The Hermite interpolatory mask $b_1$ in Theorem 4.2 is given by

$$b_1(-1) = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{bmatrix}, \qquad b_1(0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad b_1(1) = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{bmatrix},$$

with $b_1(\beta) = 0$ for $\beta \neq -1, 0, 1$. By Theorem 3.1, we have

$$\tilde{y}_0^T = [1, 0], \qquad \tilde{y}_1^T = [0, 1/15], \qquad \tilde{y}_2^T = [1/15, 0],$$
$$\tilde{y}_3^T = [0, 1/315], \qquad \tilde{y}_4^T = [1/560, 0], \qquad \tilde{y}_5^T = [0, 1/15120],$$
$$\tilde{y}_6^T = [1/37800, 0], \qquad \tilde{y}_7^T = [0, 1/1247400], \qquad \tilde{y}_8^T = [1/3991680, 0],$$
$$\tilde{y}_9^T = [0, 1/155675520].$$

By using the CBC algorithm, we find a dual mask $\tilde{a}$ of $b_1$ given by

$$\tilde{a}(0) = \begin{bmatrix} \frac{205151}{147456} & 0 \\ 0 & \frac{473515}{294912} \end{bmatrix}, \qquad \tilde{a}(1) = \begin{bmatrix} \frac{3505}{8192} & -\frac{26671}{221184} \\ \frac{30153}{8192} & -\frac{163379}{147456} \end{bmatrix},$$

$$\tilde{a}(2) = \begin{bmatrix} -\frac{13229}{73728} & \frac{11537}{221184} \\ -\frac{127889}{98304} & \frac{75973}{147456} \end{bmatrix}, \qquad \tilde{a}(3) = \begin{bmatrix} \frac{591}{8192} & -\frac{217}{8192} \\ \frac{3027}{8192} & -\frac{2333}{16384} \end{bmatrix},$$

$$\tilde{a}(4) = \begin{bmatrix} -\frac{531}{32768} & \frac{187}{16384} \\ -\frac{5109}{65536} & \frac{3721}{65536} \end{bmatrix}, \qquad \tilde{a}(\beta) = 0 \quad \forall \beta > 4,$$

and $\tilde{a}(-\beta) = U\tilde{a}(\beta)\, U$ for all $\beta \in \mathbb{N}$. Note that $\phi^{b_1} = (\phi_1^{b_1}, \phi_2^{b_1})^T$ is the well known cubic Hermite splines (see [23]) and can be explicitly expressed as

$$\phi_1^{b_1}(t) = (t+1)^2\,(-2t+1)\,\chi_{[-1,0]} + (t-1)^2\,(2t+1)\,\chi_{[0,1]}$$

and

$$\phi_2^{b_1}(t) = (t+1)^2\, t\chi_{[-1,0]} + (t-1)^2\, t\chi_{[0,1]},$$

where $\chi_E$ is the characteristic function of the set $E$. It is evident that $\phi^{b_1}$ has accuracy order 4 (see [23]) and $\nu_p(\phi^{b_1}) = 2 + 1/p$ for any $1 \leqslant p \leqslant \infty$. It is easy to check that $\phi^{\tilde{a}}$ is a dual function vector of $\phi^{b_1}$. Moreover, $\nu_2(\phi^{\tilde{a}}) \approx 1.501802$ and $\phi^{\tilde{a}}$ has accuracy order 4. Therefore, $\phi^{\tilde{a}}$ is a $C^1$ dual function vector of the well known cubic Hermite splines and $\phi^{\tilde{a}}$ is supported on $[-4, 4]$. A continuous dual function vector of $\phi^{b_1}$ with support $[-2, 2]$ was given in Dahmen *et al.* [8].

Another dual mask $\tilde{a}_1$ of $b_1$ is given by

$$\tilde{a}_1(0) = \begin{bmatrix} \frac{11046301}{7864320} & 0 \\ 0 & \frac{801731}{524288} \end{bmatrix}, \qquad \tilde{a}_1(1) = \begin{bmatrix} \frac{3235271}{7864320} & -\frac{120589}{983040} \\ \frac{20044101}{5242880} & -\frac{5864319}{5242880} \end{bmatrix},$$

$$\tilde{a}_1(2) = \begin{bmatrix} -\frac{4372675}{25165824} & \frac{35593}{786432} \\ -\frac{62155111}{41943040} & \frac{516945}{1048576} \end{bmatrix}, \qquad \tilde{a}_1(3) = \begin{bmatrix} \frac{2604317}{31457280} & -\frac{261013}{10485760} \\ \frac{5880671}{10485760} & -\frac{1801093}{10485760} \end{bmatrix},$$

$$\tilde{a}_1(4) = \begin{bmatrix} -\frac{424669}{15728640} & \frac{39523}{3145728} \\ -\frac{1785087}{10485760} & \frac{93245}{1048576} \end{bmatrix}, \qquad \tilde{a}_1(5) = \begin{bmatrix} \frac{183239}{31457280} & -\frac{57119}{31457280} \\ \frac{267049}{10485760} & -\frac{82729}{10485760} \end{bmatrix},$$

$$\tilde{a}_1(6) = \begin{bmatrix} -\frac{195121}{125829120} & \frac{1051}{1048576} \\ -\frac{285911}{41943040} & \frac{9}{2048} \end{bmatrix}, \qquad \tilde{a}_1(\beta) = 0 \quad \forall \beta > 6,$$

and $\tilde{a}_1(-\beta) = U\tilde{a}_1(\beta)\, U$ for all $\beta \in \mathbb{N}$. It is not difficult to verify that $\phi^{\tilde{a}_1}$ is a dual function vector of $\phi^{b_1}$ such that $\phi^{\tilde{a}_1}$ has accuracy order 10 and $\nu_2(\phi^{\tilde{a}_1}) \approx 2.12273$. Therefore, $\nu_\infty(\phi^{\tilde{a}_1}) > 1.62272$ which implies $\phi^{\tilde{a}_1} \in C^1$.

EXAMPLE 4.6. The Hermite interpolatory mask $b_2$ in Theorem 4.2 is given by

$$b_2(-3) = \begin{bmatrix} \frac{13}{512} & \frac{5}{512} \\ -\frac{3}{512} & -\frac{1}{512} \end{bmatrix}, \qquad b_2(-1) = \begin{bmatrix} \frac{243}{512} & \frac{405}{512} \\ -\frac{81}{512} & -\frac{81}{512} \end{bmatrix},$$

$$b_2(0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad b_2(1) = \begin{bmatrix} \frac{243}{512} & -\frac{405}{512} \\ \frac{81}{512} & -\frac{81}{512} \end{bmatrix},$$

$$b_2(3) = \begin{bmatrix} \frac{13}{512} & -\frac{5}{512} \\ \frac{3}{512} & -\frac{1}{512} \end{bmatrix}, \qquad b_2(\beta) = 0 \quad \forall \beta \neq -3, -1, 0, 1, 3.$$

By Theorem 3.1, we have $\tilde{y}_0^T = [1, 0]$. A dual mask $\tilde{a}$ of $b_2$ is given by

$$\tilde{a}(0) = \begin{bmatrix} \frac{1157}{1024} & 0 \\ 0 & \frac{1747}{1024} \end{bmatrix}, \qquad \tilde{a}(1) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{5}{2} & -\frac{9}{8} \end{bmatrix}, \qquad \tilde{a}(2) = \begin{bmatrix} -\frac{7}{128} & \frac{11}{128} \\ -\frac{455}{2048} & \frac{411}{1024} \end{bmatrix},$$

$$\tilde{a}(3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{a}(4) = \begin{bmatrix} -\frac{21}{2048} & \frac{5}{1024} \\ -\frac{215}{4096} & \frac{51}{2048} \end{bmatrix}, \qquad \tilde{a}(\beta) = 0 \quad \forall \beta > 4,$$

and $\tilde{a}(-\beta) = U\tilde{a}(\beta)\, U$ for all $\beta \in \mathbb{N}$. Then $\phi^{b_2}$ is a Hermite interpolant such $\nu_2(\phi^{b_2}) \approx 3.394956$ and $\phi^{b_2}$ has accuracy order 8. $\phi^{\tilde{a}}$ is a dual function vector of $\phi^{b_2}$ with $\nu_2(\phi^{\tilde{a}}) \approx 0.75620$ and has accuracy order 1. Therefore, $\phi^{\tilde{a}}$ is a continuous dual function vector of $\phi^{b_2} \in C^2$ since $\nu_\infty(\phi^{\tilde{a}}) \geqslant 0.25620$.

EXAMPLE 4.7. The Hermite interpolatory mask $a$ by modifying the mask $b_2$ is given by

$$a(-3) = \begin{bmatrix} \frac{13}{512} & \frac{1}{32} \\ -\frac{3}{512} & -\frac{1}{128} \end{bmatrix}, \qquad a(-1) = \begin{bmatrix} \frac{243}{512} & \frac{27}{32} \\ -\frac{81}{512} & -\frac{27}{128} \end{bmatrix},$$

$$a(0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad a(1) = \begin{bmatrix} \frac{243}{512} & -\frac{27}{32} \\ \frac{81}{512} & -\frac{27}{128} \end{bmatrix},$$

$$a(3) = \begin{bmatrix} \frac{13}{512} & -\frac{1}{32} \\ \frac{3}{512} & -\frac{1}{128} \end{bmatrix}, \qquad a(\beta) = 0 \quad \forall \beta \neq -3, -1, 0, 1, 3.$$

Then $\tilde{y}_0^T = [1, 0]$ and $\tilde{y}_1^T = [0, 5/56]$. By the CBC algorithm, a dual mask $\tilde{a}$ of $a$ is given by

$$\tilde{a}(0) = \begin{bmatrix} \frac{2395}{2048} & 0 \\ 0 & \frac{3311}{2048} \end{bmatrix}, \qquad \tilde{a}(1) = \begin{bmatrix} \frac{1}{2} & -\frac{27}{128} \\ \frac{5453}{2160} & -\frac{149}{160} \end{bmatrix},$$

$$\tilde{a}(2) = \begin{bmatrix} -\frac{161}{2048} & \frac{321}{4096} \\ -\frac{176519}{552960} & \frac{33341}{92160} \end{bmatrix}, \qquad \tilde{a}(3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{a}(4) = \begin{bmatrix} -\frac{25}{4096} & \frac{21}{8192} \\ -\frac{7741}{221184} & \frac{2771}{184320} \end{bmatrix}, \qquad \tilde{a}(\beta) = 0 \quad \forall \beta > 4,$$

and $\tilde{a}(-\beta) = U\tilde{a}(\beta)\, U$ for all $\beta \in \mathbb{N}$. Then $\phi^a$ is a Hermite interpolant such that $\nu_2(\phi^a) \approx 3.84745$ and $\phi^a$ has accuracy order 7. $\phi^{\tilde{a}}$ is a dual function vector of $\phi^a$ such that $\nu_2(\phi^{\tilde{a}}) \approx 0.91843$ and $\phi^{\tilde{a}}$ has accuracy order 2. Therefore, $\phi^{\tilde{a}}$ is a continuous dual function vector of $\phi^a$ since $\nu_\infty(\phi^{\tilde{a}}) \geqslant 0.41843$. Moreover, $\phi^a \in C^3$.

Though the primal masks in all the above examples are interpolatory masks, the CBC algorithm can be easily applied to the general case. Let us take an example from Plonka and Strela [37] as the primal mask and then we use the CBC algorithm to construct a dual function vector for it.

EXAMPLE 4.8. The primal mask a in [37] is given by

$$a(0)=\begin{bmatrix}\frac{13}{64} & -\frac{9}{64}\\ \frac{11}{64} & -\frac{7}{64}\end{bmatrix},\qquad a(1)=\begin{bmatrix}\frac{51}{64} & -\frac{9}{64}\\ \frac{21}{64} & \frac{9}{64}\end{bmatrix}\qquad a(2)=\begin{bmatrix}\frac{51}{64} & \frac{9}{64}\\ -\frac{21}{64} & \frac{9}{64}\end{bmatrix},$$

$$a(3)=\begin{bmatrix}\frac{13}{64} & \frac{9}{64}\\ -\frac{11}{64} & -\frac{7}{64}\end{bmatrix},\qquad a(\beta)=0\quad \forall\beta\neq 0,\,1,\,2,\,3.$$

Then as pointed out in [37] that the refinable function vector $\phi^a$ is a piecewise polynomial B-spline of order 6 with double knots. $v_p(\phi^a)=4+1/p$ for any $1\leqslant p\leqslant\infty$ and $\phi^a$ has accuracy order 6. By Theorem 3.1, we have

$$\tilde{y}_0^T=[1,0],\qquad \tilde{y}_1^T=[3/2,\,-3/14],\qquad \tilde{y}_2^T=[17/14,\,-9/28],$$

$$\tilde{y}_3^T=[39/56,\,-43/168],\qquad \tilde{y}_4^T=[529/1680,\,-1/7].$$

By using the CBC algorithm, we find a dual mask $\tilde{a}$ of $a$ given by

$$\tilde{a}(2)=\begin{bmatrix}\frac{159239}{122880} & \frac{28291}{122880}\\ -\frac{12198157}{3317760} & \frac{919531}{3317760}\end{bmatrix},\qquad \tilde{a}(3)=\begin{bmatrix}-\frac{18621}{40960} & \frac{7653}{40960}\\ \frac{5400373}{3317760} & -\frac{1043381}{3317760}\end{bmatrix},$$

$$\tilde{a}(4)=\begin{bmatrix}\frac{1347}{5120} & -\frac{1273}{10240}\\ -\frac{847007}{1105920} & -\frac{47627}{368640}\end{bmatrix},\qquad \tilde{a}(5)=\begin{bmatrix}-\frac{1169}{7680} & -\frac{551}{31720}\\ \frac{961993}{3317760} & \frac{331813}{3317760}\end{bmatrix},$$

$$\tilde{a}(6)=\begin{bmatrix}\frac{411}{8192} & \frac{531}{8192}\\ -\frac{321}{4096} & -\frac{63}{512}\end{bmatrix},\qquad \tilde{a}(7)=\begin{bmatrix}-\frac{19}{8192} & -\frac{45}{8192}\\ -\frac{5}{4096} & \frac{3}{1024}\end{bmatrix},$$

and $\tilde{a}(\beta)=U\tilde{a}(3-\beta)\,U$ for $\beta=-4,\ -3,\ -2,\ -1,\ 0,\ 1$, and otherwise, $\tilde{a}(\beta)=0$. Then $\phi^{\tilde{a}}$ is a dual function vector of $\phi^a$ with $v_2(\phi^{\tilde{a}})\approx 1.13543$ and has accuracy order 5. Therefore, $\phi^{\tilde{a}}$ is a continuous dual function vector of $\phi^a$ since $v_\infty(\phi^{\tilde{a}})\geqslant 0.63543$.

The graphs of all the above examples in this section are presented in Figs. 1–5. More examples of biorthogonal multiwavelets can be constructed by using the CBC algorithm studied in this paper and such algorithm can be easily implemented. Finally, we mention that the study of approximation order of biorthogonal multiwavelets in this paper is not only useful for construction of biorthogonal multiwavelets but also helpful for construction of orthogonal multiwavelets.

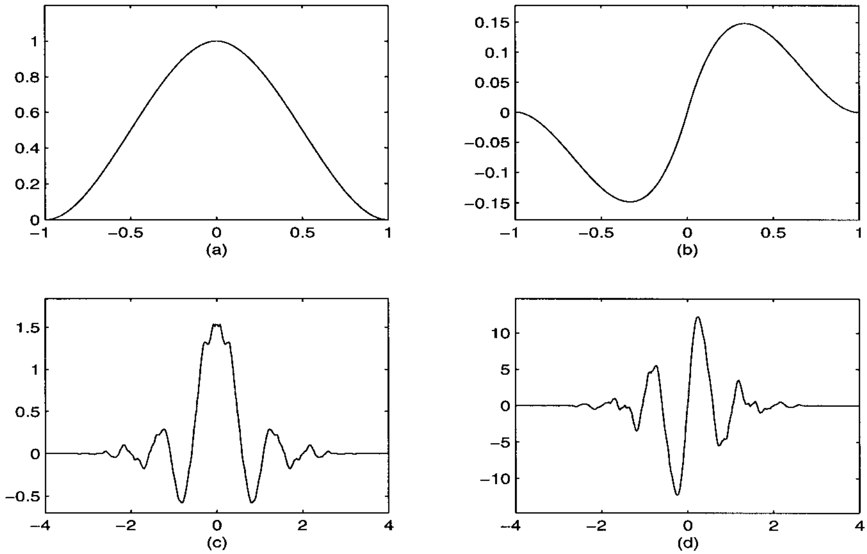



**FIG. 1.** (a), (b), (c), and (d). Graphs of $\phi_1^{b_1}$, $\phi_2^{b_1}$, $\phi_1^{\tilde{a}}$, and $\phi_2^{\tilde{a}}$ in Example 4.5, respectively. $\phi^{b_1}$ is the piecewise Hermite cubics and $\phi^{\tilde{a}}$ is a $C^1$ dual function vector of $\phi^{b_1}$.

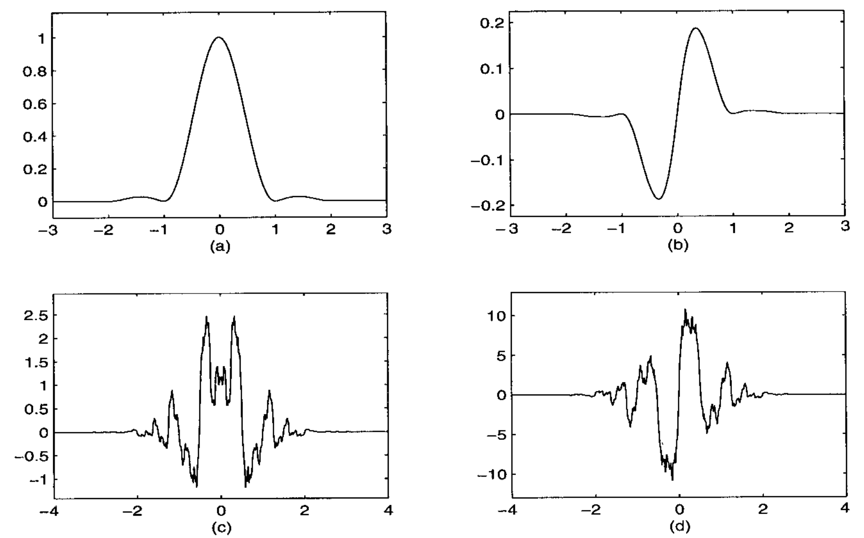



**FIG. 2.** (a), (b), (c), and (d). Graphs of $\phi_1^{b_2}$, $\phi_2^{b_2}$, $\phi_1^{\tilde{a}}$, and $\phi_2^{\tilde{a}}$ in Example 4.6, respectively. $\phi^{b_2}$ is a $C^2$ Hermite interpolant and $\phi^{\tilde{a}}$ is a continuous dual function vector of $\phi^{b_2}$.

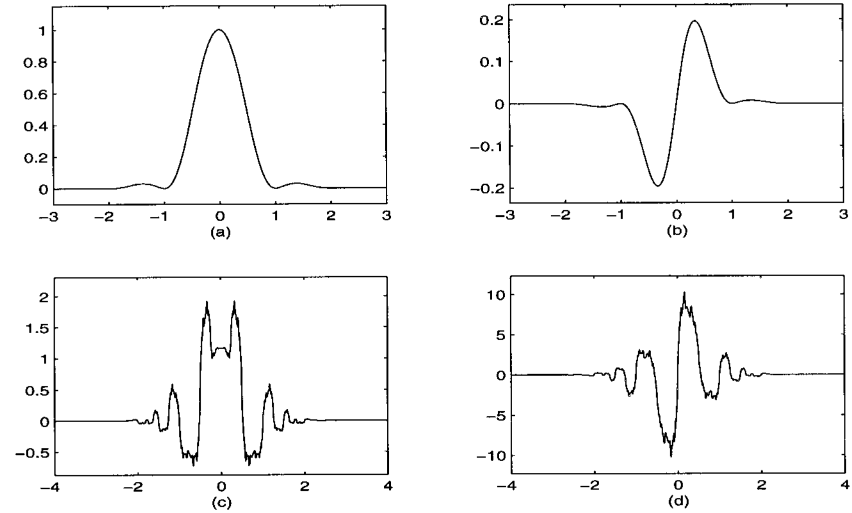



**FIG. 3.** (a), (b), (c), and (d). Graphs of $\phi_1^a$, $\phi_2^a$, $\phi_1^{\tilde{a}}$, and $\phi_2^{\tilde{a}}$ in Example 4.7, respectively. $\phi^a$ is a $C^3$ Hermite interpolant and $\phi^{\tilde{a}}$ is a continuous dual function vector of $\phi^a$.

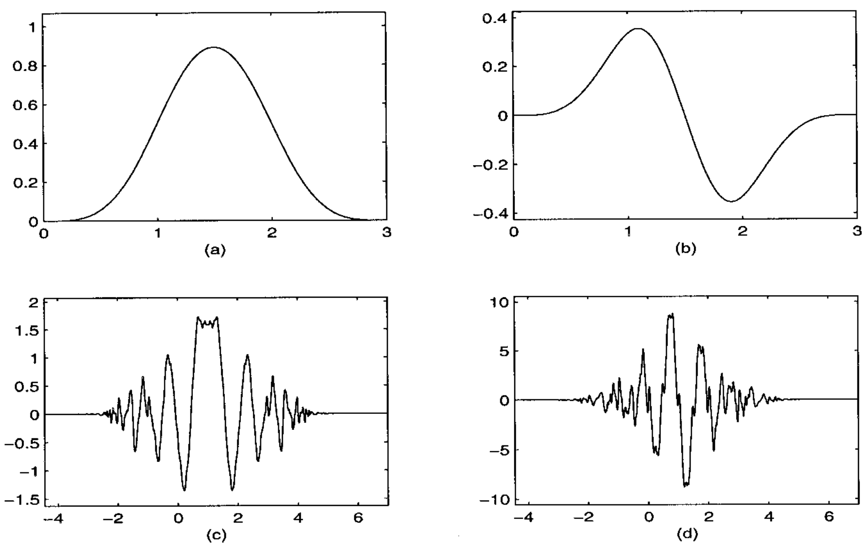
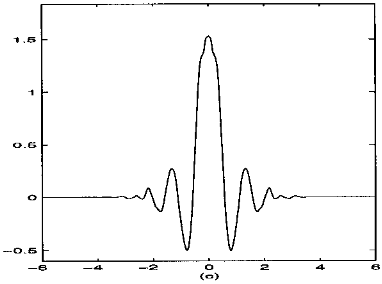



**FIG. 4.** (a), (b), (c), and (d). Graphs of $\phi_1^a$, $\phi_2^a$, $\phi_1^{\tilde{a}}$, and $\phi_2^{\tilde{a}}$ in Example 4.8, respectively. $\phi^a \in C^3$ is a polynomial B-spline of order 6 with double knots and $\phi^{\tilde{a}}$ is a continuous dual function vector of $\phi^a$.



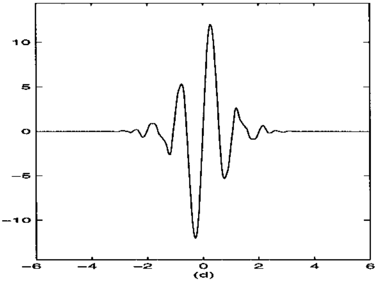




**FIG. 5.** (c) Graph of $\phi_1^{\tilde{a}_1}$. (d) Graph of $\phi_2^{\tilde{a}_1}$ in Example 4.5. $\phi^{\tilde{a}_1} \in C^1$ is a dual function vector of $\phi^{b_1}$.

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